2

Topics:

1. Ideas to construct the mean electro-motive force without calculations
2. Estimation of turbulent effects

Mean-field equations

$$
\begin{aligned}
\partial_t \langle \boldsymbol{B} \rangle &= \boldsymbol{\nabla} \times (\boldsymbol{\mathcal{E}} + \langle \boldsymbol{U} \rangle \times \langle \boldsymbol{B} \rangle), \\
\frac{\partial}{\partial t} \bar{\rho} r^2 \sin^2\theta \Omega &= -\boldsymbol{\nabla} \cdot \Bigg(r \sin\theta \Bigg(\bar{\rho} \hat{\mathbf{T}}_\phi + r \sin\theta \Omega \langle \boldsymbol{U} \rangle - \frac{\langle \boldsymbol{B} \rangle \langle B_\phi \rangle}{4\pi} \Bigg) \Bigg), \hat{\mathbf{T}}_\phi = \hat{T}_{\mathrm{ij}} e^j_\varphi \\
\bar{\rho} \bar{T} \left(\frac{\partial \langle s \rangle}{\partial t} + (\langle \boldsymbol{U} \rangle \cdot \boldsymbol{\nabla}) \langle s \rangle \right) &= -\boldsymbol{\nabla} \cdot (\mathbf{F}^c + \mathbf{F}^{\mathrm{rad}}) - \hat{T}_{\mathrm{ij}} \frac{\partial \langle U_i \rangle}{\partial r_j} - \boldsymbol{\mathcal{E}} \cdot \boldsymbol{\nabla} \times \langle \boldsymbol{B} \rangle
\end{aligned}
$$

Effects of the turbulent flows and magnetic fields:

$$
\begin{aligned}
\boldsymbol{\mathcal{E}} &= \langle \boldsymbol{u} \times \boldsymbol{b} \rangle, \\
\hat{T}_{\mathrm{ij}} &= \langle u_i u_j \rangle - \frac{1}{4\pi\bar{\rho}} (\langle b_i b_j \rangle - \delta_{\mathrm{ij}} \langle b^2 \rangle) \\
F_i^c &= -c_p \bar{\rho} \bar{T} \kappa_{\mathrm{ij}} \nabla_j \langle s \rangle, \kappa_{\mathrm{ij}} = \langle u_i u_j \rangle
\end{aligned}
$$

$\mathcal{E} = \langle \boldsymbol{u} \times \boldsymbol{b} \rangle$

We have to solve:

$$\begin{aligned}
\partial_t \boldsymbol{b} &= \boldsymbol{\nabla} \times (\boldsymbol{u} \times \langle \boldsymbol{B} \rangle + \langle \boldsymbol{U} \rangle \times \boldsymbol{b}) + \eta \nabla^2 \boldsymbol{b} + \nabla \times (\boldsymbol{u} \times \boldsymbol{b} - \mathcal{E}) + \mathfrak{G} \\
\bar{\rho} \partial_t u_i &= 2\bar{\rho}(\boldsymbol{u} \times \boldsymbol{\Omega})_i - \nabla_i \left(p + \frac{(\boldsymbol{b} \cdot \langle \boldsymbol{B} \rangle)}{8\pi} \right) + \nu \Delta \bar{\rho} u_i + f_i + \mathfrak{F}_i \\
&- \nabla_j (\bar{\rho} u_i \langle U_j \rangle + u_j \langle U_i \rangle) + \nabla_j (\bar{\rho} T_{\mathrm{ij}} - \bar{\rho} \hat{T}_{\mathrm{ij}}) \\
&+ \frac{1}{4\pi} \nabla_j (b_i \langle B_j \rangle + b_j \langle B_i \rangle)
\end{aligned}$$

Problems:

- Nonlinearity results to a closure problem: to resolve $\boldsymbol{b}$ evolution we have to know $\boldsymbol{u} \times \boldsymbol{b}$, and same for $\boldsymbol{u}$
- Let us try to guess the general expression for $\langle \boldsymbol{u} \times \boldsymbol{b} \rangle$

Reflection symmetry

Reflection symmetry is a fundamental property of the basic physical laws. This means that the MHD laws do not change under reflection at arbitrary point about arbitrary plan. This is so called parity conservation.

Meanwhile some of the quantities and mathematical operators depend on orientation of the coordinate frame, e.g.:

Operator	Geometric meaning
curl, $\nabla\times$	closed directional loop
vector product, $\times$	oriented area
scalar-vector product	oriented volume

of three vectors

Properties of the reflection symmetries, for arbitrary vectors: $\mathbf{a}, \mathbf{b}, \mathbf{c}$

$$\begin{aligned} \mathbf{a}^{ref} &= -\mathbf{a},\ \mathbf{a}^{ref} \times \mathbf{b}^{ref} = a \times b, \\ \nabla \times \mathbf{a}^{ref} &= \nabla \times \mathbf{a}, \\ (\mathbf{a}^{ref} \times \mathbf{b}^{ref}) \cdot \mathbf{c}^{ref} &= -(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \end{aligned}$$

Parity(P) symmetry of the basic quantities

Using the Maxwell equations

$$\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = -\nabla\times\mathbf{E},\ \ \nabla\cdot\mathbf{B}=0$$
$$\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \nabla\times\mathbf{B}-\frac{4\pi}{c}\mathbf{j},\ \ \nabla\cdot\mathbf{E}=4\pi\rho$$

consider reflection transformation about the origin $\mathbf{x}=0$:

$$\mathbf{u}^{ref}(\mathbf{x},t) = -\mathbf{u}(-\mathbf{x},t),\ \mathbf{E}^{ref}(\mathbf{x},t)=-\mathbf{E}(-\mathbf{x},t),$$
$$\mathbf{J}^{ref}(\mathbf{x},t) = -\mathbf{J}(-\mathbf{x},t),\ \mathbf{B}^{ref}(\mathbf{x},t)=\mathbf{B}(-\mathbf{x},t)$$

In the "old school" physics it is common to call vectors like $\mathbf{u}$, $\mathbf{E}$, and $\mathbf{J}$ as the polar vector (normal vector, or simply vector) and the vectors like $\mathbf{b}$, $\mathbf{B}$, and $\mathbf{A} = \nabla^{-1} \times \mathbf{B}$ are called - axial vectors (pseudo-vectors). Therefore,

- The mean electromotive force : $\mathcal{E} = \langle \mathbf{u} \times \mathbf{b} \rangle$ is a vector
- $\mathbf{u} \cdot \nabla \times \mathbf{u}$ (kinetic helicity), $\mathbf{b} \cdot \nabla \times \mathbf{b}$ (current helicity), $\mathbf{u} \cdot \mathbf{b}$ (cross-helicity) and $\mathbf{A} \cdot \mathbf{B}$ (magnetic helicity) are all pseudo-scalars
- Sequence is as follows: 0-rank: scalar is P invariant and pseudo-scalar changes sign; 1-rank: pseudo-vector is P invariant; 2-rank tensor is P invariant, pseudo-tensor isn't.

Expression for $\mathcal{E}$

From induction equation:

$$\frac{\partial \mathbf{b}}{\partial t} = \nabla \times (\langle \boldsymbol{U} \rangle \times \mathbf{b} + \mathbf{u} \times \langle \boldsymbol{B} \rangle + \mathbf{u} \times \mathbf{b} - \langle \mathbf{u} \times \mathbf{b} \rangle)$$

- we see that, in general we can assume the functional relation:

$$\mathbf{b} \sim \mathbf{b}(\mathbf{u}, \langle \boldsymbol{U} \rangle, \langle \boldsymbol{B} \rangle)$$

and $\mathbf{b}$ as a linear to $\langle \boldsymbol{B} \rangle$ (if we don't take effects of $\langle \boldsymbol{B} \rangle$ on $\boldsymbol{u}$)

- Assume $\partial_x \boldsymbol{b} \sim \frac{\boldsymbol{b}}{\ell_c}$ and $\partial_x \langle \boldsymbol{B} \rangle \sim \frac{\langle \boldsymbol{B} \rangle}{L_a}$, and the scale separation $\ell_c \ll L_a$,

- Then the set of vectors to construct $\langle \mathbf{u} \times \mathbf{b} \rangle$ may consists of,

   i. the large-scale field: $\langle \boldsymbol{B} \rangle, \nabla \times \langle \boldsymbol{B} \rangle$ and perhaps $\boldsymbol{\Omega}, \nabla \times \langle \boldsymbol{U} \rangle$, $\boldsymbol{\nabla} \bar{\rho}$

   ii. the average effects of the turbulent fields: $\boldsymbol{\nabla} \langle u^{(0)2} \rangle$, $\boldsymbol{\nabla} \langle b^{(0)2} \rangle$, etc.

Expression for $\mathcal{E}$

Consider the isotropic background turbulence, to satisfy parity we put:

$$\begin{aligned}\boldsymbol{\mathcal{E}} &= \langle \mathbf{u}\times\mathbf{b}\rangle = \alpha\langle \boldsymbol{B}\rangle + \mathbf{V}\times\langle \boldsymbol{B}\rangle - \eta_T\nabla\times\langle \boldsymbol{B}\rangle \\ &+ \Gamma\boldsymbol{\Omega} + \delta^{\Omega}\left(\boldsymbol{\Omega}\times\nabla\times\langle \boldsymbol{B}\rangle\right) + \delta^{W}\left(\left(\nabla\times\langle \boldsymbol{U}\rangle\right)\times\nabla\times\langle \boldsymbol{B}\rangle\right) + \\ &\quad o\left(\frac{\ell_c}{L_a}\right)\end{aligned} \tag{1}$$

Here α and Γ should be pseudo-scalar and η, δ^{Ω}, δ^{W} -usual scalars.

The α effect is a turbulent generation effect and η is a turbulent diffusion. They can be estimated from solution of equations for fluctuating field.

$\mathbf{V}$ is turbulent pumping. How we guessed it? In tensor notation the first line of Eq(1) can be rewritten in more general and shorter form:

The most basic $\mathcal{E}$

$$\mathcal{E}_i = a_{ij}\langle B_j\rangle + \eta_{ijn}\nabla_j\langle B_n\rangle + o\left(\frac{\ell}{L}\right)$$

Here, we used the standard notation for summation about the repeated indices. Any a_{ij} can be decomposed

$$a_{ij} \equiv \frac{1}{2}(a_{ij} + a_{ji}) + \frac{1}{2}(a_{ij} - a_{ji})$$

The most simple symmetric tensor is the Kronecker delta symbol: $\delta_{ij} = \delta_{ji}$. The most simple antisymmetric tensor is the Levi-Chevitta symbol ε_{ijn} , which is antisymmetric about odd number permutation of all indices:

$$\delta_{ij}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\varepsilon_{1jn}=\begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix},\varepsilon_{2jn}=\begin{pmatrix}0&0&-1\\0&0&0\\1&0&0\end{pmatrix},\varepsilon_{3jn}=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}$$

Put, $\frac{1}{2}(\alpha_{ij}-\alpha_{ji})=\varepsilon_{inj}V_n$, and $\eta_{ijn}=\eta_T\varepsilon_{ijn}$. Therefore, in following reflection symmetry rules we rewrite: $\mathcal{E}_i=\alpha\delta_{ij}\langle B_j\rangle+\varepsilon_{inj}V_n\langle B_j\rangle+\eta_T\varepsilon_{inj}\nabla_j\langle B_n\rangle$ which is identical to

$$\boldsymbol{\mathcal{E}}=\alpha\langle\boldsymbol{B}\rangle+\mathbf{V}\times\langle\boldsymbol{B}\rangle-\eta_T\nabla\times\langle\boldsymbol{B}\rangle.$$

Calculation

$$\partial_t \boldsymbol{b} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \langle \boldsymbol{B} \rangle) + \eta \nabla^2 \boldsymbol{b} + \nabla \times (\boldsymbol{u} \times \boldsymbol{b} - \mathcal{E}) + \mathfrak{G}$$

$$\bar{\rho}\partial_t u_i = 2\bar{\rho}(\boldsymbol{u} \times \boldsymbol{\Omega})_i - \nabla_i \left(p + \frac{(\boldsymbol{b} \cdot \langle \boldsymbol{B} \rangle)}{8\pi} \right) + \nu \Delta \bar{\rho} u_i + f_i + \mathfrak{F}_i$$

$$+ \nabla_j (\bar{\rho} T_{\mathrm{ij}} - \bar{\rho} \hat{T}_{\mathrm{ij}}) + \frac{1}{4\pi} \nabla_j (b_i \langle B_j \rangle + b_j \langle B_i \rangle)$$

Comsider the linear effects of $\langle \boldsymbol{B} \rangle$ on the backround turbulence

so $\boldsymbol{b} \longrightarrow \boldsymbol{b} + \boldsymbol{b}^{(0)}$ and $\boldsymbol{u} \longrightarrow \boldsymbol{u} + \boldsymbol{u}^{(0)}$ after subtraction the background turbulence MHD equations we get

$$\partial_t \boldsymbol{b} = \boldsymbol{\nabla} \times (\boldsymbol{u}^{(0)} \times \langle \boldsymbol{B} \rangle) + \eta \nabla^2 \boldsymbol{b} + \nabla \times (\boldsymbol{u}^{(0)} \times \boldsymbol{b} + \boldsymbol{u} \times \boldsymbol{b}^{(0)} - \mathcal{E})$$

$$\bar{\rho}\partial_t u_i = 2\bar{\rho}(\boldsymbol{u}^{(0)} \times \boldsymbol{\Omega})_i - \nabla_i \left(p + \frac{((\boldsymbol{b} + \boldsymbol{b}^{(0)}) \cdot \langle \boldsymbol{B} \rangle)}{8\pi} \right) + \nu \Delta \bar{\rho} u_i +$$

$$+ \quad \nabla_j(\bar{\rho}T_{\mathrm{ij}} - \bar{\rho}\hat{T}_{\mathrm{ij}}) + \frac{1}{4\pi}\nabla_j\Big(b_i^{(0)}\langle B_j\rangle + b_j^{(0)}\langle B_i\rangle\Big)$$

High Reynolds number limit

We apply ideas taken from the mixing-length approximation and the dimensional analysis. It is interesting to consider two cases:

I.The large Reynolds and magnetic Reynolds numbers. In this case we neglect the effect of microscopic diffusivity η and replace the induction term in equation for fluctuating magnetic field by following one:

$$\partial_t \boldsymbol{b} - \nabla \times (\boldsymbol{u}^{(0)} \times \boldsymbol{b} + \boldsymbol{u} \times \boldsymbol{b}^{(0)} - \mathcal{E}) \to \frac{\boldsymbol{b}}{\tau_c}$$

$$\frac{\boldsymbol{b}}{\boldsymbol{\tau_c}} = [(\langle \boldsymbol{B} \rangle \cdot \boldsymbol{\nabla}) \boldsymbol{u}^{(0)} - (\boldsymbol{u}^{(0)} \cdot \boldsymbol{\nabla}) \langle \boldsymbol{B} \rangle] + \frac{\boldsymbol{b}^{(0)}}{\boldsymbol{\tau_c}},$$

$$\frac{\boldsymbol{u}}{\boldsymbol{\tau_c}} = 2\boldsymbol{u}^{(0)} \times \boldsymbol{\Omega} - \boldsymbol{\nabla}\left(\frac{(\langle \boldsymbol{B}\rangle \cdot \boldsymbol{b}^{(0)})}{4\pi\bar{\rho}}\right) + \frac{(\langle \boldsymbol{B}\rangle \cdot \boldsymbol{\nabla})}{4\pi\bar{\rho}}\boldsymbol{b}^{(0)} + \frac{(\boldsymbol{b}^{(0)} \cdot \boldsymbol{\nabla})}{4\pi\bar{\rho}}\langle \boldsymbol{B}\rangle + \frac{\boldsymbol{u}^{(0)}}{\boldsymbol{\tau_c}} + \textbf{shear} + \textbf{NL}(\langle \boldsymbol{B}\rangle)\cdots$$

Note, that the pressure fluctuations are inside of $\frac{u^{(0)}}{\tau_c}$. The effect of magnetic pressure, $-\boldsymbol{\nabla}(\langle \boldsymbol{B}\rangle \cdot b^{(0)})/4\pi\bar{\rho}$, can be ignored, as well, for $\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$. The turbulent electromotive force is

$$\mathcal{E}_i = \varepsilon_{ijn}\left(\langle u_j^{(0)} b_n\rangle + \langle u_j b_n^{(0)}\rangle\right)$$

Calculation of $\mathcal{E}$, Re,Rm»1

$$
\begin{aligned}
\mathcal{E}_i &= \varepsilon_{\rm ijn}\, u_j^{(0)}\big[(\langle \boldsymbol{B}\rangle\cdot\nabla)u_n^{(0)} - (\boldsymbol{u}^{(0)}\cdot\nabla)\langle B_n\rangle\big]\tau_c + \varepsilon_{\rm ijn}\, u_j^{(0)} b_n^{(0)} \\
&+ 2\tau_c\varepsilon_{\rm ijn}\,(\boldsymbol{u}^{(0)}\times\Omega)_j b_n^{(0)} + \tau_c\varepsilon_{\rm ijn}\frac{(\langle \boldsymbol{B}\rangle\cdot\nabla)}{4\pi\bar\rho} b_j^{(0)} b_n^{(0)} \\
&+ \tau_c\varepsilon_{\rm ijn}\frac{(\boldsymbol{b}^{(0)}\cdot\nabla)}{4\pi\bar\rho}\langle B_j\rangle b_n^{(0)} + \tau_c\varepsilon_{\rm ijn}\, u_j^{(0)} b_n^{(0)}
\end{aligned}
$$

Let us consider the isotropic turbulence : $\left\langle u_i^{(0)} u_j^{(0)}\right\rangle = \frac{1}{3}\delta_{ij}\left\langle u^{(0)2}\right\rangle$,and same for the magnetic fluctuations: $\left\langle b_i^{(0)} b_j^{(0)}\right\rangle = \frac{1}{3}\delta_{ij}\left\langle b^{(0)2}\right\rangle$. Collecting the same type of terms we get

$$
\mathcal{E}_i = a_{ip}\langle B_p\rangle + \eta_{inp}\nabla_n\langle B_p\rangle + 2\tau_c\langle \boldsymbol{u}^{(0)}\cdot\boldsymbol{b}^{(0)}\rangle\Omega_i,
$$

where we assume that $\langle \boldsymbol{u}^{(0)} \times \boldsymbol{b}^{(0)} \rangle = 0$ and, perhaps, $\langle \boldsymbol{u}^{(0)} \cdot \boldsymbol{b}^{(0)} \rangle \neq 0$(Yoshizawa &Yokoi 1990 ; Brandenburg & Raedler 2008).

Re,Rm»1, turbulent diffusion

We have,

$$\mathcal{E}_i = a_{ip}\langle B_p\rangle + \eta_{inp}\nabla_n\langle B_p\rangle + 2\tau_c\langle \boldsymbol{u}^{(0)}\cdot\boldsymbol{b}^{(0)}\rangle\Omega_i,$$

where we assume that $\langle \boldsymbol{u}^{(0)} \times \boldsymbol{b}^{(0)}\rangle = 0$

$$\begin{aligned} a_{\mathrm{ip}} &= \tau_c\varepsilon_{ijn}\Big\langle u_j^{(0)}\nabla_p u_n^{(0)}\Big\rangle + \tau_c\varepsilon_{ijn}\frac{\Big\langle b_n^{(0)}\nabla_p b_j^{(0)}\Big\rangle}{4\pi\rho} \\ \eta_{\mathrm{inp}} &= -\frac{\tau_c}{3}\left(\Big\langle u^{(0)2}\Big\rangle + \frac{3}{8\pi\bar{\rho}}\Big\langle b^{(0)2}\Big\rangle\right)\varepsilon_{\mathrm{inp}} \end{aligned}$$

The tensor η_{inp} is the turbulent diffusion, because $\varepsilon_{\mathrm{inp}}\nabla_n\langle B_p\rangle \equiv (\nabla\times\langle\boldsymbol{B}\rangle)_i$. The magnetic fluctuations of the background turbulence, $\left\langle b^{(0)2} \right\rangle$, which can stem from the small-scale dynamo, amplify the turbulent diffusion.

The alpha effect and pumping

For the case of the isotropic background turbulence. The only possible general view expression of $\mathcal{E}$ is

$$\boldsymbol{\mathcal{E}} = \langle \mathbf{u} \times \mathbf{b} \rangle = \alpha \langle \mathbf{B} \rangle + \Gamma\, \boldsymbol{\Omega} - \eta_T \nabla \times \langle \mathbf{B} \rangle + \text{pumping}(\nabla \langle u^{(0)2} \rangle, \langle \nabla b^{(0)2} \rangle) \ldots$$

We already got $\Gamma = 2\tau_c \langle \boldsymbol{u}^{(0)} \cdot \boldsymbol{b}^{(0)} \rangle$ and $\eta_T = \frac{\tau_c}{3} \Big(\langle u^{(0)2} \rangle + \frac{3}{8\pi\bar{\rho}} \langle b^{(0)2} \rangle \Big)$

The alpha effect and turbulent pumping can be formally rewritten as follows

$$\begin{aligned} \mathcal{E}_i &= a_{\rm ip} \langle B_p \rangle = \frac{1}{2}(a_{\rm ip} + a_{\rm pi}) \langle B_p \rangle + \frac{1}{2}(a_{\rm ip} - a_{\rm pi}) \langle B_p \rangle \\ &= \alpha \delta_{\rm ip} \langle B_p \rangle + \varepsilon_{\rm inp} V_n \langle B_p \rangle \end{aligned}$$

Where, to get the last term, we use $(a_{\mathrm{ip}} - a_{\mathrm{pi}}) = \varepsilon_{\mathrm{inp}}\varepsilon_{\mathrm{nml}}a_{\mathrm{ml}}$, and $V_n = -\frac{1}{2}\varepsilon_{\mathrm{nml}}a_{\mathrm{ml}}$. So, to get pumping we have to calculate $V_n = -\frac{1}{2}\varepsilon_{\mathrm{nml}}a_{\mathrm{ml}}$. The direct calculation of α can be tedious. Instead, ..

The kinetic part of α

We have

$$\begin{aligned}
\mathcal{E}_i &= \varepsilon_{\rm ijn}\, u_j^{(0)}\big[(\langle \boldsymbol{B}\rangle\cdot\nabla)u_n^{(0)} - (\boldsymbol{u}^{(0)}\cdot\nabla)\langle B_n\rangle\big]\tau_c + \varepsilon_{\rm ijn}\, u_j^{(0)} b_n^{(0)} \\
&+ 2\tau_c\varepsilon_{\rm ijn}\,(\boldsymbol{u}^{(0)}\times\Omega)_j b_n^{(0)} + \tau_c\varepsilon_{\rm ijn}\frac{(\langle \boldsymbol{B}\rangle\cdot\nabla)}{4\pi\bar{\rho}} b_j^{(0)} b_n^{(0)} \\
&+ \tau_c\varepsilon_{\rm ijn}\frac{(\boldsymbol{b}^{(0)}\cdot\nabla)}{4\pi\bar{\rho}}\langle B_j\rangle b_n^{(0)} + \tau_c\varepsilon_{\rm ijn}\, u_j^{(0)} b_n^{(0)}
\end{aligned}$$

Denote:

$$w_{\rm jpn} = \tau_c\big\langle u_j^{(0)}\nabla_p u_n^{(0)}\big\rangle$$

for the kinetic part of the α effect in the Cartesian coordinates for isotropic α we get,

$$\mathcal{E}_1 = B_1(w_{213} - w_{312}), \mathcal{E}_2 = B_2(w_{321} - w_{123}), \mathcal{E}_3 = B_3(w_{132} - w_{231})$$

Using isotropy condition we arrive to $(w_{213} - w_{312}) = (w_{321} - w_{123}) = (w_{132} - w_{231})$:

$$\begin{aligned} \alpha^{(K)} &= \frac{1}{3}(w_{213} - w_{312} + w_{321} - w_{123} + w_{132} - w_{231}) \\ &= -\frac{1}{3}(w_{123} - w_{132} + w_{231} - w_{213} + w_{312} - w_{321}) \end{aligned}$$

In Cartesian coordinates $\mathbf{u}^{(0)} \cdot \nabla \times \mathbf{u}^{(0)} = u_1^{(0)} \nabla_2 u_3^{(0)}$-$u_1^{(0)} \nabla_3 u_2^{(0)} = w_{123} - w_{132}$.

Thus we have:

$$\alpha^{(K)} = -\frac{\tau_c}{3} \langle \mathbf{u}^{(0)} \cdot \nabla \times \mathbf{u}^{(0)} \rangle$$

Intepretation of the kinetc α effect

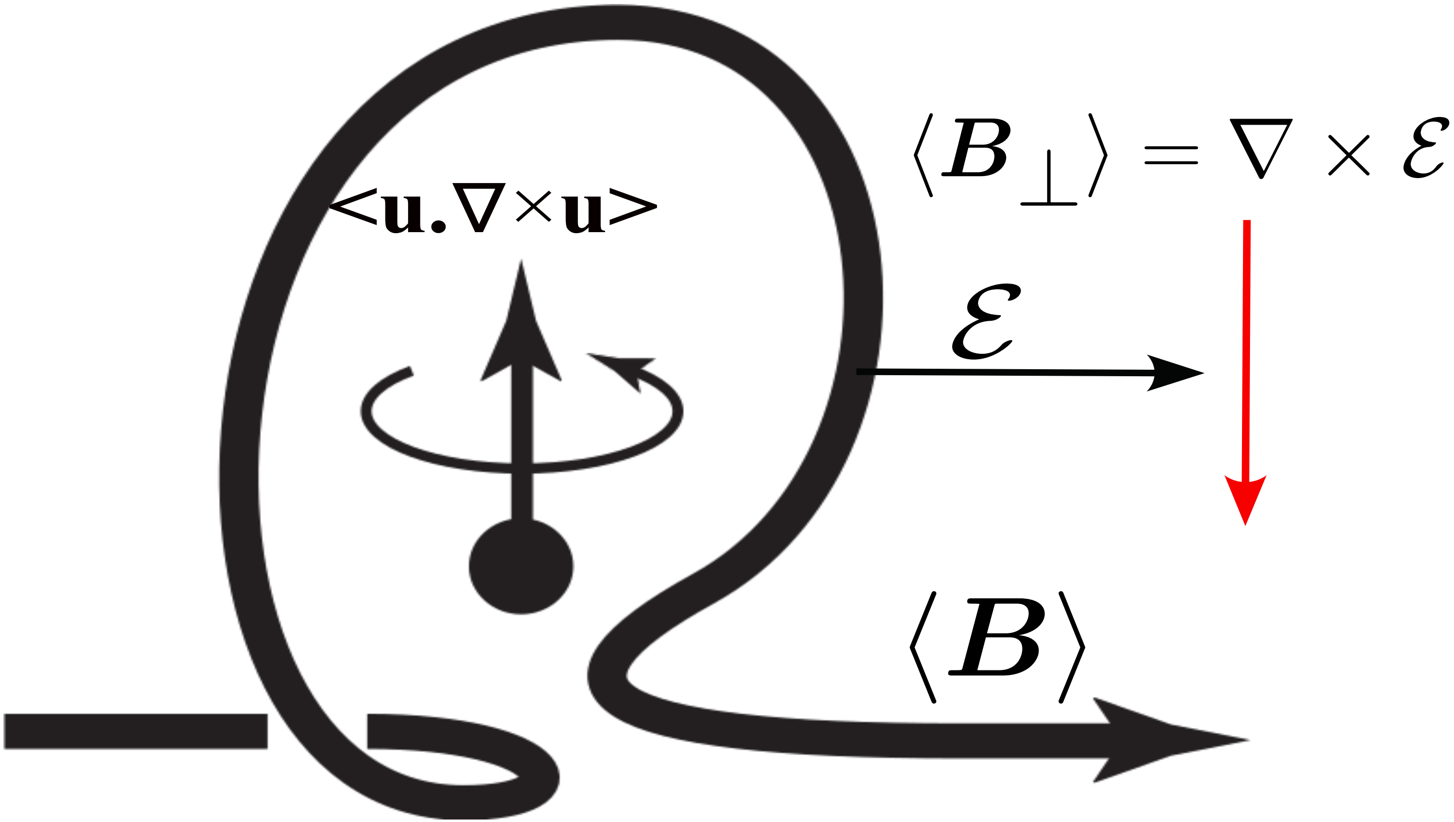
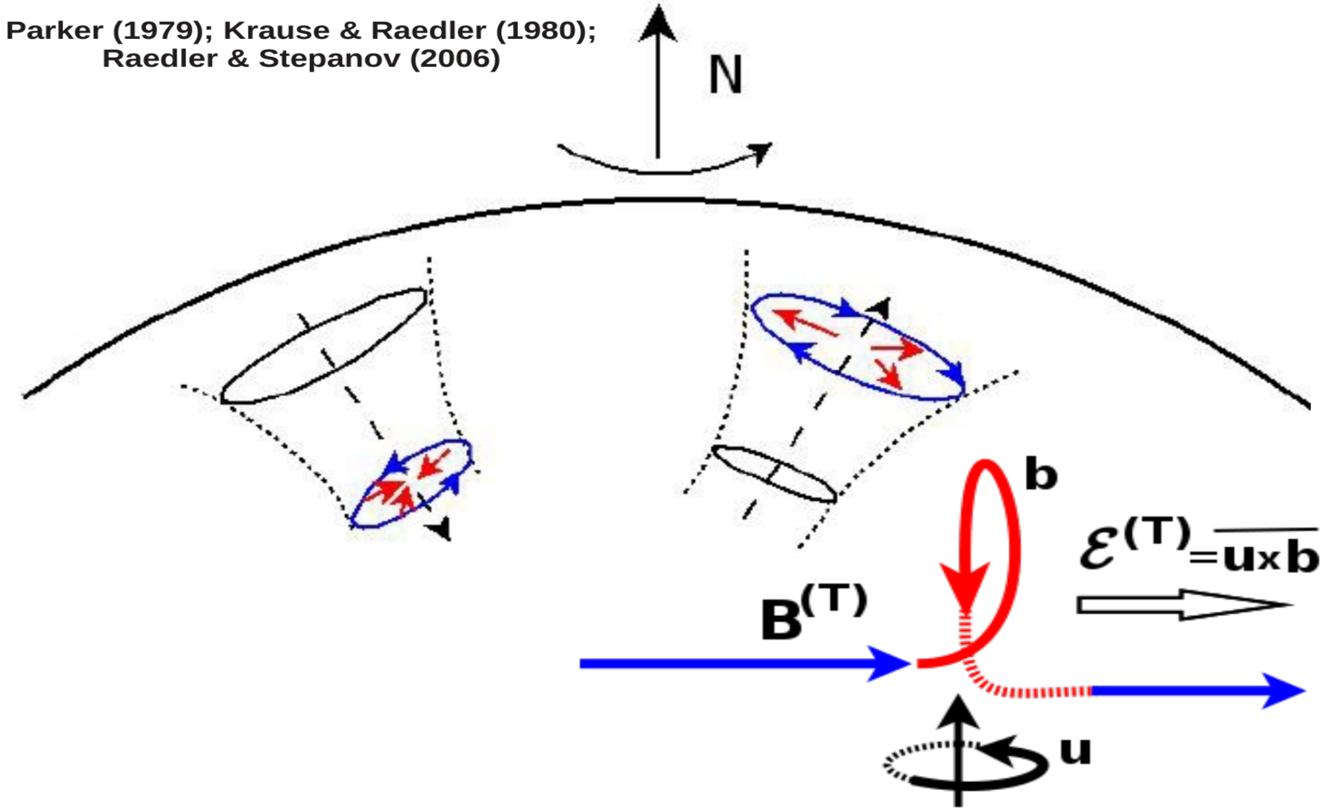


Figure 1. Generation of $\mathcal{E}$ align to $\langle B \rangle$ by means of the kinetic helicity, Krause and Raedler (1980)

The α effect by means of rotation and stratification



Parker (1979); Krause & Raedler (1980);
Raedler & Stepanov (2006)
N
b
$\mathcal{E}^{(T)} = \overline{\mathbf{u} \times \mathbf{b}}$
$\mathbf{B}^{(T)}$
u

Figure 2.

Algebraic α quenching

Magnetic field prevents motions across and quenches effects by factor B^{-1} in each direction, Proctor (2007)

Figure 3.

$$\alpha \sim -\frac{\tau_c}{3}\frac{\langle \mathbf{u}^{(0)} \cdot \nabla \times \mathbf{u}^{(0)} \rangle}{1+\left(\frac{B}{B_{\rm eq}}\right)^2}$$

The magnetic part of α

$$\mathcal{E}_i^M = \frac{\tau_c}{4\pi\rho}\varepsilon_{\mathrm{ijn}}\left\langle b_n^{(0)}\nabla_p b_j^{(0)}\right\rangle\langle B_p\rangle$$

The same calculation in Cartesian coordinates for the magnetic part, $w_{\mathrm{npj}} = \tau_c\frac{\left\langle b_n^{(0)}\nabla_p b_j^{(0)}\right\rangle}{4\pi\rho}$:

$$\mathcal{E}_1 = B_1(w_{312} - w_{213}), \mathcal{E}_2 = B_2(w_{123} - w_{321}), \mathcal{E}_3 = B_3(w_{231} - w_{132})$$

Using isotropy condition we arrive to $(w_{312} - w_{213}) = (w_{123} - w_{321}) = (w_{231} - w_{132})$:

$$h^{(C)} \;\; = \;\; \frac{1}{3}(w_{312} - w_{213} + w_{123} - w_{321} + w_{231} - w_{132})$$

$$= \frac{1}{3}(w_{123} - w_{132} + w_{231} - w_{213} + w_{312} - w_{321})$$

$$\alpha^{(M)} = \frac{\tau_c}{12\pi\bar{\rho}}\langle \mathbf{b}^{(0)} \cdot \nabla \times \mathbf{b}^{(0)} \rangle$$

So, the total alpha effect reads (Frisch, Leorat & Pouqet 1976):

$$\boldsymbol{\mathcal{E}}^{\alpha} = -\frac{\tau_c}{3}\left(\langle \mathbf{u}^{(0)} \cdot \nabla \times \mathbf{u}^{(0)} \rangle - \frac{\langle \mathbf{b}^{(0)} \cdot \nabla \times \mathbf{b}^{(0)} \rangle}{4\pi\bar{\rho}} \right)\langle \boldsymbol{B} \rangle$$

The nonlinear α effect

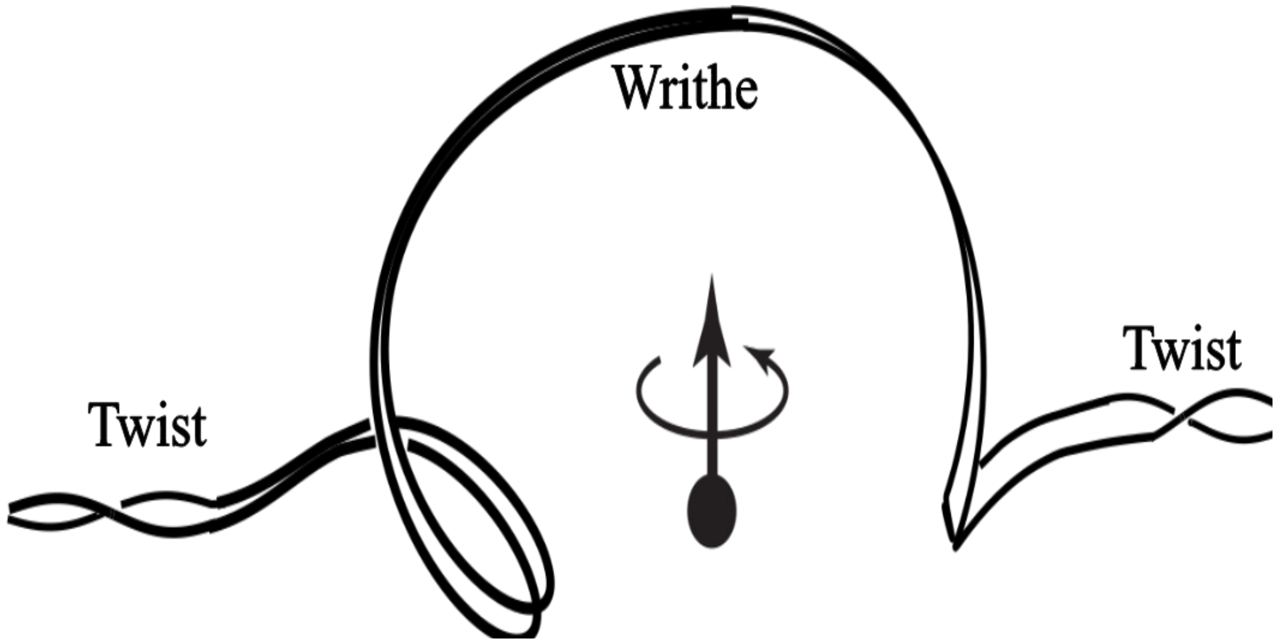


Figure 4. Bi-helical magnetic field Blackman & Brandenburg (2001)

$$\alpha \sim -\frac{\tau_c}{3}\frac{\langle \mathbf{u}^{(0)} \cdot \nabla \times \mathbf{u}^{(0)} \rangle}{1+\left(\frac{B}{B_{\rm eq}}\right)^2} + \tau_c \frac{\langle \mathbf{b}^{(0)} \cdot \nabla \times \mathbf{b}^{(0)} \rangle}{12\pi\bar{\rho}}$$

Re,Rm»1, turbulent pumping

The turbulent pumping is the skew symmetric part of

$$a_{\mathrm{ip}} = \tau_c \varepsilon_{\mathrm{ijm}} \left\langle u_j^{(0)} \nabla_p u_m^{(0)} \right\rangle + \tau_c \varepsilon_{\mathrm{ijm}} \frac{\left\langle b_m^{(0)} \nabla_p b_j^{(0)} \right\rangle}{4\pi\rho}$$

To get pumping velocity we have to compute $V_n = -\frac{1}{2}\varepsilon_{\mathrm{nip}} a_{\mathrm{ip}}$. Note, that

$$\varepsilon_{\mathrm{nip}} \varepsilon_{\mathrm{ijm}} = \varepsilon_{\mathrm{ipn}} \varepsilon_{\mathrm{ijm}} = \left(\delta_{\mathrm{pj}} \delta_{\mathrm{nm}} - \delta_{\mathrm{pm}} \delta_{\mathrm{nj}}\right)$$

Then

$$\varepsilon_{\mathrm{nip}} \varepsilon_{\mathrm{ijm}} \tau_c \left\langle u_j^{(0)} \nabla_p u_m^{(0)} \right\rangle = (\delta_{\mathrm{pj}} \delta_{\mathrm{nm}} - \delta_{\mathrm{pm}} \delta_{\mathrm{nj}}) \tau_c \left\langle u_j^{(0)} \nabla_p u_m^{(0)} \right\rangle =$$

$$= \tau_c \left\langle u_p^{(0)} \nabla_p u_n^{(0)} \right\rangle - \left\langle u_n^{(0)} \nabla_p u_p^{(0)} \right\rangle = \tau_c \nabla_p \left\langle u_p^{(0)} u_n^{(0)} \right\rangle$$

$$\tau_c \langle u_p^{(0)} \nabla_p u_n^{(0)} \rangle \;=\; \frac{\tau_c}{3} \nabla_p \delta_{\rm pn} \langle u^{(0)2} \rangle$$

where we employ conditions for isotropic turbulence identity, $\langle u_p^{(0)} u_n^{(0)} \rangle = \frac{1}{3} \delta_{\rm pn} \langle u^{(0)2} \rangle$.

Repeating this for the magnetic part we get

$$V_n = -\frac{1}{2} \nabla_n \left(\eta_T - \frac{\tau_c}{3} \frac{\langle b^{(0)2} \rangle}{4\pi\bar{\rho}} \right)$$

If there is equi-partition between the kinetic and magnetic fluctuations of the background turbulence then

$$\langle u^{(0)2} \rangle \;=\; \frac{\langle b^{(0)2} \rangle}{4\pi\bar{\rho}},$$

$$\boldsymbol{V}^{(\rm eff)} = 0$$

$\boldsymbol{V} = -\frac{1}{2}\boldsymbol{\nabla}\eta_T$ is the diamagnetic pumping discovered by Zeldovich(1956)

Diamagnetic pumping

$$\boldsymbol{\mathcal{E}} = \boldsymbol{V}^{(\mathrm{eff})} \times \langle \boldsymbol{B} \rangle, \boldsymbol{V}^{(\mathrm{eff})} = -\frac{1}{2} \boldsymbol{\nabla} \eta_T$$

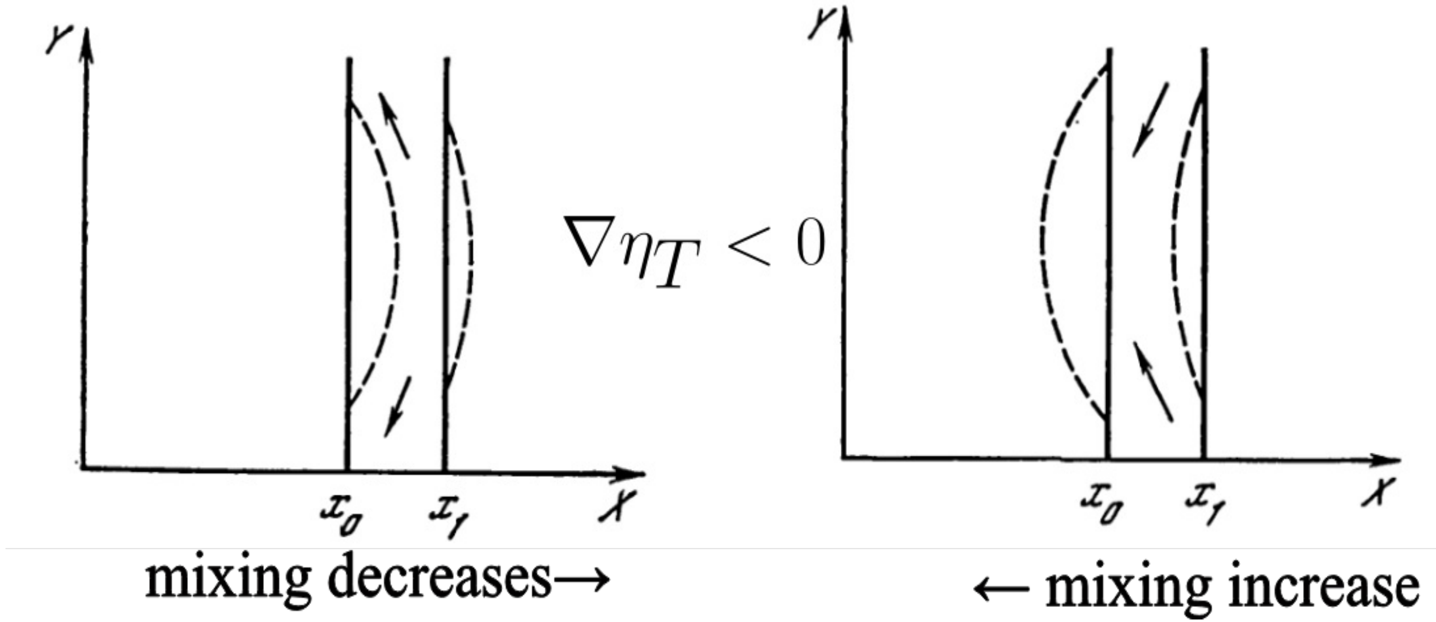


Figure 5. Diamagnetic pumping because of the nonuniform turbulent mixing (Zeldovich 1956), picture from Vainshtain (1980). On Figure, the turbulent mixing decrease outward, therefore the effective pumping work outward.

Re,Rm»1, $\mathcal{E}$ (overview)

Finally, the mean electromotive force for the case Re,Rm»1,the isotropic background turbulence, and if $\langle \mathbf{u}^{(0)} \cdot \boldsymbol{b}^{(0)} \rangle \neq 0$,

$$
\begin{aligned}
\boldsymbol{\mathcal{E}} = \langle \mathbf{u} \times \mathbf{b} \rangle &= \alpha \langle \mathbf{B} \rangle + \Gamma \, \boldsymbol{\Omega} + \boldsymbol{V}^{(\mathrm{eff})} \times \langle \boldsymbol{B} \rangle - \eta_T \nabla \times \langle \mathbf{B} \rangle, \\
\alpha &= -\frac{\tau_c}{3} \left(\langle \mathbf{u}^{(0)} \cdot \nabla \times \mathbf{u}^{(0)} \rangle - \frac{\langle \mathbf{b}^{(0)} \cdot \nabla \times \mathbf{b}^{(0)} \rangle}{4\pi\bar{\rho}} \right), \\
\Gamma &= \tau_c \langle \mathbf{u}^{(0)} \cdot \boldsymbol{b}^{(0)} \rangle, \\
\boldsymbol{V}^{(\mathrm{eff})} &= -\frac{1}{2} \boldsymbol{\nabla} \left(\eta_T - \frac{\tau_c}{3} \frac{\langle b^{(0)2} \rangle}{4\pi\bar{\rho}} \right), \\
\eta_T &= \frac{\tau_c}{3} \left(\langle u^{(0)2} \rangle + \frac{3}{8\pi\bar{\rho}} \langle b^{(0)2} \rangle \right)
\end{aligned}
$$

Magnetic fluctuations try to cancel the turbulent generation and pumping effects and increase the turbulent diffusion.

Low Reynolds number limit

II. The small magnetic Reynolds number. In this case, the evolution of **b** is defined by the magnetic diffusion time $t_\eta = \frac{\ell^2}{\eta}$, and similar for the turbulent flow,i.e.,

$$\boldsymbol{b} = \frac{\ell^2}{\eta}[(\langle \boldsymbol{B}\rangle\cdot\nabla)\boldsymbol{u}^{(0)} - (\boldsymbol{u}^{(0)}\cdot\nabla)\langle \boldsymbol{B}\rangle] + \frac{\ell^2}{\eta}\boldsymbol{b}^{(0)}$$

$$\boldsymbol{u} = \frac{2\ell^2}{\nu}\boldsymbol{u}\times\boldsymbol{\Omega} - \frac{2\ell^2}{\nu}\nabla\left(\frac{(\langle \boldsymbol{B}\rangle\cdot\boldsymbol{b}^{(0)})}{4\pi\bar{\rho}}\right) + \frac{2\ell^2}{\nu}\frac{(\langle \boldsymbol{B}\rangle\cdot\nabla)}{4\pi\bar{\rho}}\boldsymbol{b}^{(0)} + \frac{2\ell^2}{\nu}\frac{(\boldsymbol{b}^{(0)}\cdot\nabla)}{4\pi\bar{\rho}}\langle \boldsymbol{B}\rangle + \frac{2\ell^2}{\nu}\boldsymbol{u}^{(0)} + \mathbf{shear} + \mathbf{NL}(\langle \boldsymbol{B}\rangle)\cdots$$

The turbulent electromotive force is

$$\mathcal{E}_i = \varepsilon_{\mathrm{ijn}}\left(\left\langle u_j^{(0)} b_n \right\rangle + \left\langle u_j b_n^{(0)} \right\rangle\right)$$

We can use the results obtained for the limit Rm,Re»1 and substitute τ_c by $\tau_\nu = \frac{2\ell^2}{\nu}$

Re,Rm≪1 , $\mathcal{E}$

$$\boldsymbol{\mathcal{E}} = \langle \mathbf{u} \times \mathbf{b} \rangle = \alpha \langle \mathbf{B} \rangle + \Gamma \, \boldsymbol{\Omega} + \boldsymbol{V}^{(\mathrm{eff})} \times \langle \boldsymbol{B} \rangle - \eta_T \nabla \times \langle \mathbf{B} \rangle,$$

$$\alpha = -\frac{\mathrm{Rm}\, \tau_c}{3} \left(\langle \mathbf{u}^{(0)} \cdot \nabla \times \mathbf{u}^{(0)} \rangle - \frac{\langle \mathbf{b}^{(0)} \cdot \nabla \times \mathbf{b}^{(0)} \rangle}{4\pi \bar{\rho}} \right),$$

$$\Gamma = \tau_c \mathrm{Rm} \langle \mathbf{u}^{(0)} \cdot \boldsymbol{b}^{(0)} \rangle,$$

$$\boldsymbol{V}^{(\mathrm{eff})} = -\frac{\mathrm{Rm}}{2} \boldsymbol{\nabla} \left(\eta_T - \frac{\tau_c}{3} \frac{\langle b^{(0)2} \rangle}{4\pi \bar{\rho}} \right),$$

$$\eta_T = \frac{\mathrm{Rm} \tau_c}{3} \left(\langle u^{(0)2} \rangle + \frac{3}{8\pi \bar{\rho}} \langle b^{(0)2} \rangle \right)$$

The level of the magnetic perturbations

For the case of the large Re,Rm»1:

$$\langle \boldsymbol{b}^2 \rangle \sim \boldsymbol{\tau_c^2} \langle [(\langle \boldsymbol{B} \rangle \cdot \boldsymbol{\nabla})\boldsymbol{u}^{(0)}]^2 \rangle \sim \frac{\tau_c^2}{\ell_c^2} \langle \boldsymbol{u}^{(0)2} \rangle \langle B \rangle^2 \sim \langle B \rangle^2$$

The more accurate calculation using the turbulent spectra results to $\langle \boldsymbol{b}^2 \rangle \sim \ln(\mathrm{Rm}) \langle \boldsymbol{B} \rangle^2$, see, Rogachevskii (2021).

For the case of the large Re,Rm«1:

$$\langle \boldsymbol{b}^2 \rangle \sim \mathrm{Rm}^2 \langle \boldsymbol{B} \rangle^2$$

Summary

- The general structure of the mean electromotive force can be guessed from the reflection symmetry properties of $\boldsymbol{\mathcal{E}}\,, \boldsymbol{U}$ and $\boldsymbol{B}$.
- Using scale-separation and the semi-qualitative analysis for the isotropic background turbulence we found

$$\boldsymbol{\mathcal{E}} = \langle \mathbf{u} \times \mathbf{b} \rangle \;=\; \alpha \langle \mathbf{B} \rangle + \Gamma\, \boldsymbol{\Omega} + \boldsymbol{V}^{(\mathrm{eff})} \times \langle \boldsymbol{B} \rangle - \eta_T \nabla \times \langle \mathbf{B} \rangle,$$

generation transport dif-fusion

- The accurate estimation of α, Γ, $V^{(\mathrm{eff})}$ and η_T requires a thorough solution equations for the turbulent flows and magnetic fields:

- Double-scale Fourier, using either quasi-linear approach(SOCA, FOSA) or equations for the second order moments and τ approximation
- Test-field method (numerical solution of equation for $\boldsymbol{u}$ and $\boldsymbol{b}$ for given $\langle \boldsymbol{B} \rangle$.
- Data assimilation and approximation of $\boldsymbol{\mathcal{E}}$ from results of the global convection simulations. See further information in Brandenburg et al (SpSciRev2023)