The mean-field dynamo theory review

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L.L. Kitchatinov, G. Ruediger, A.G. Kosovichev, D.D.Sokoloff, V.N. Obridko and many others

Dynamo theory studies the nature of the cosmic magnetic fields (MF), including the origin of the solar and stellar magnetic activity, origin of the MF of planets, astrophysical planetary disks, galaxies and the Universe.

Our goals:

- Show how the dynamo theory can be deduced from the first principles governing evolution of the magnetic field in astrophysical plasma
- Describe methods employed in the theory
- Examples of application

Plan

1)Motivation and formal introduction

2)The turbulent electro-motive force

3) Dynamo scenarios

4) Differential rotation and turbulent angular momentum transport

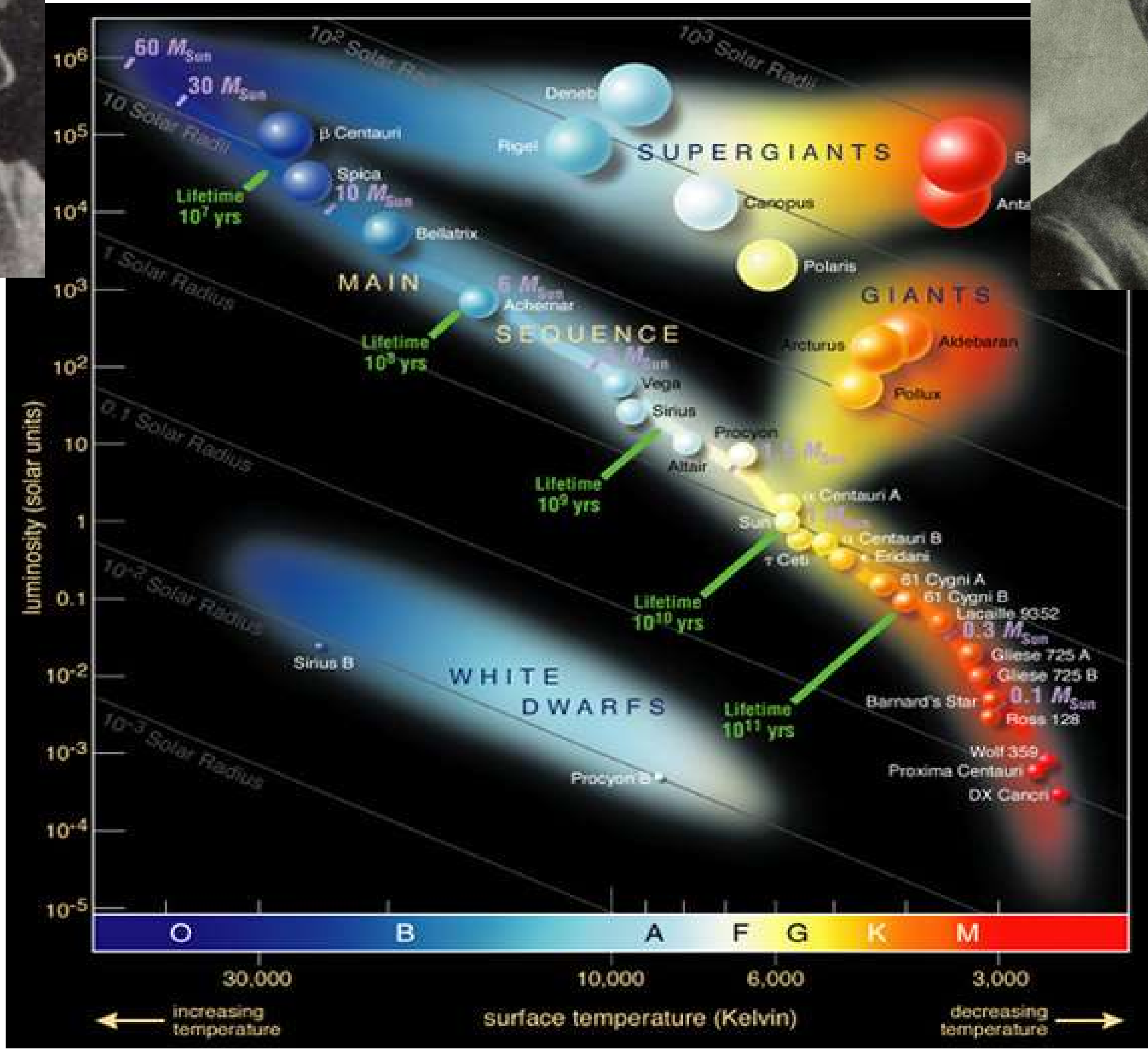
5)Mean-field solar dynamo model and nonlinear effects

Lecture I: Motivation and Formal introduction

- Observations
- Formal introduction
- Basic models for driving turbulence

Observations

The main sequence

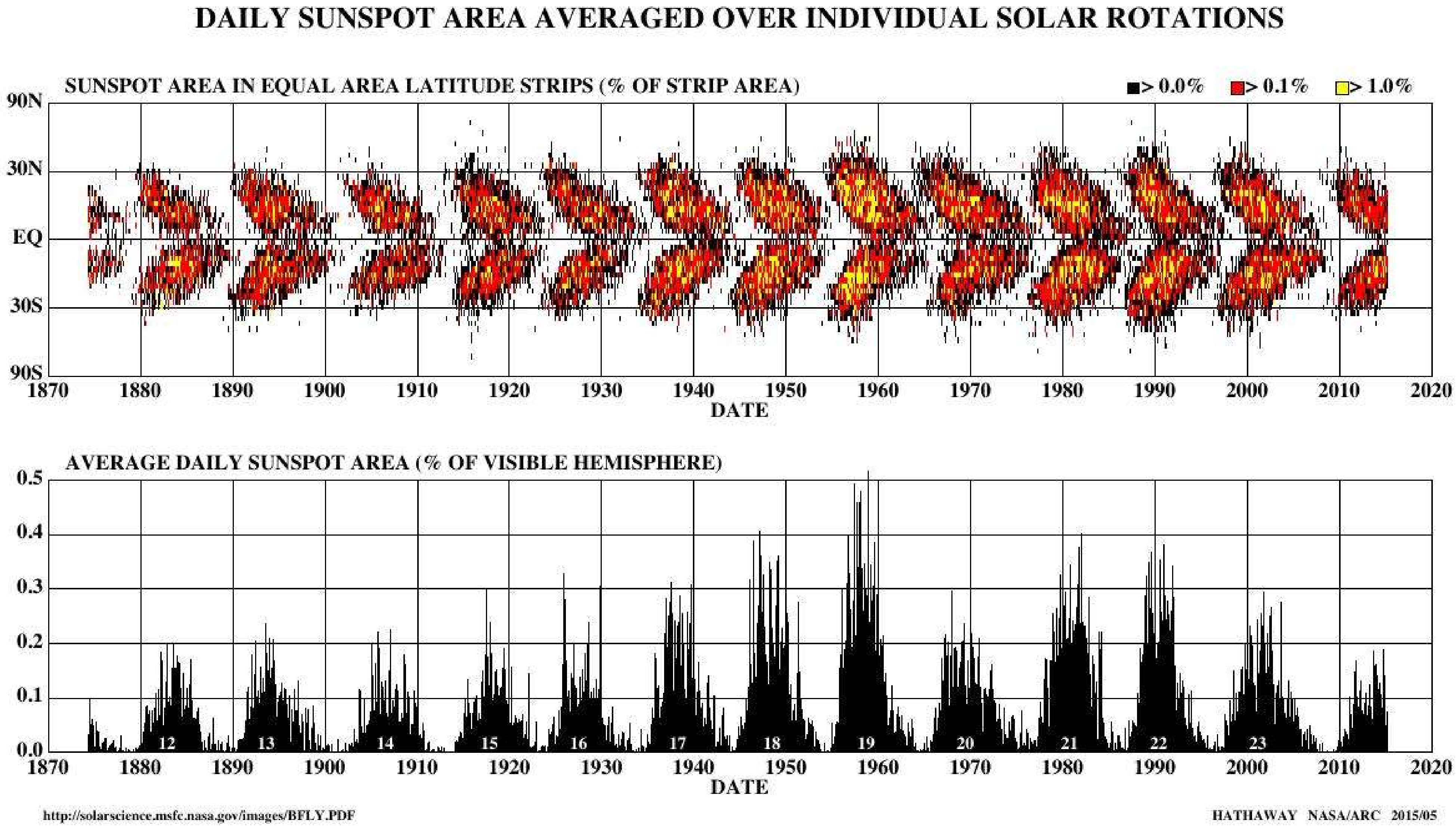


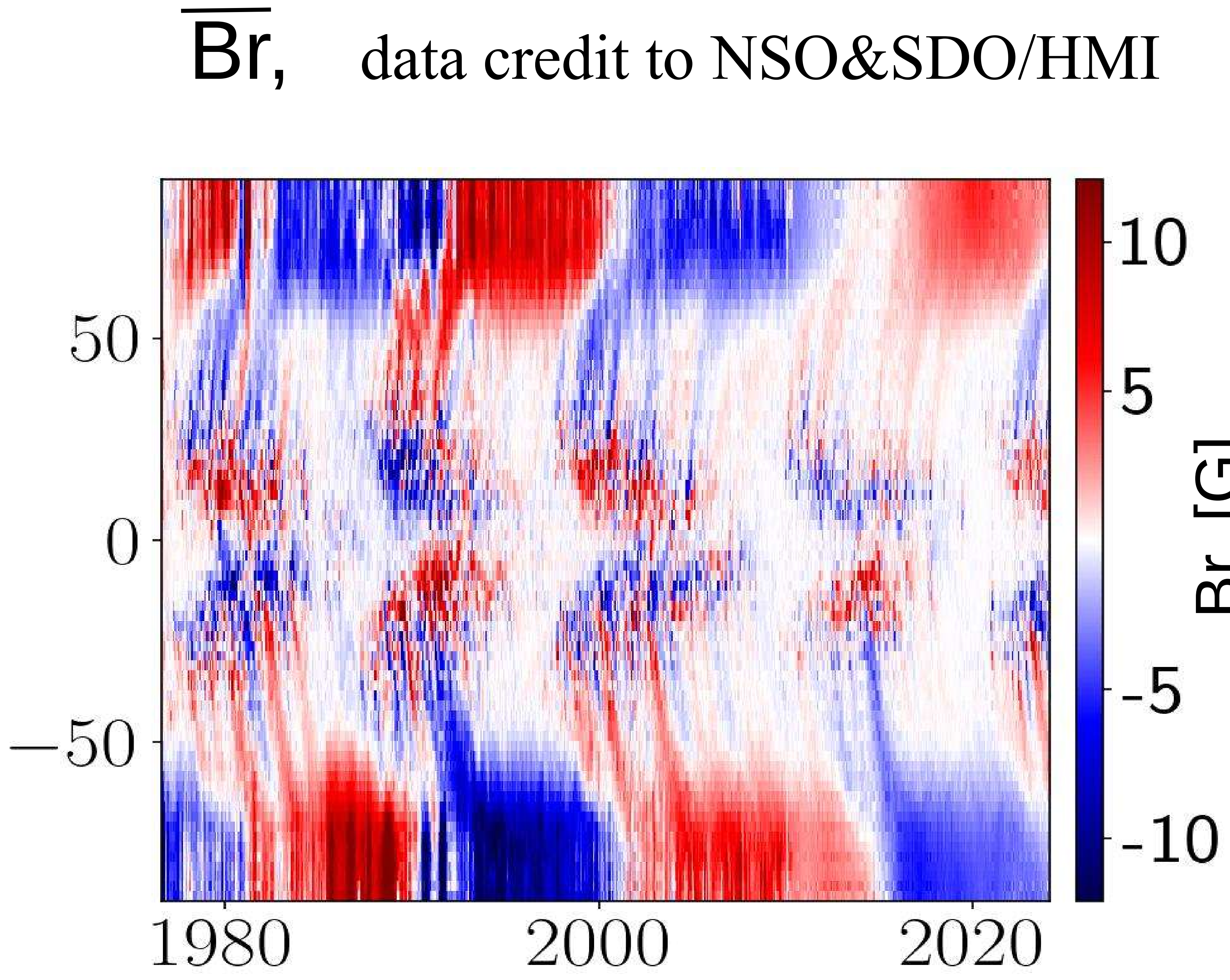
The low sequence stars have external CZ and develop the diverse variety of magnetic activity phenomena: Magnetic Cycles, Chromospheric and Coronal activity, Flares, Winds and etc

Credit to ESA

Solar cycle of global activity

Daily sunspot area, Hathaway NASA/ARC 2016/10



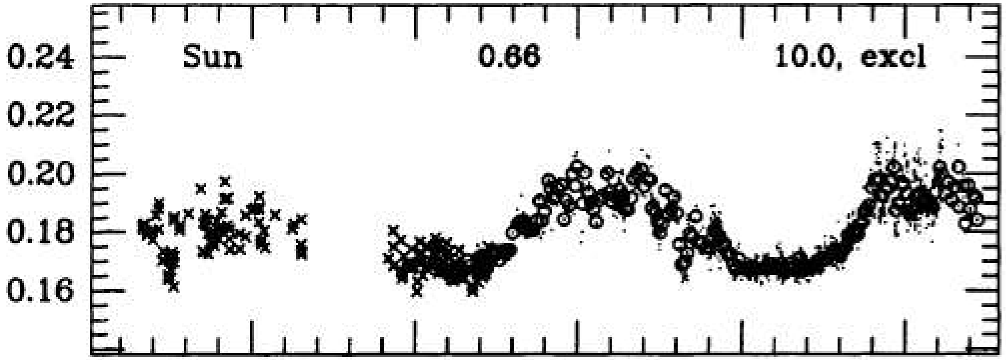


$\overline{Br}$, data credit to NSO&SDO/HMI
50
0
−50
1980
2000
2020
10
5
-5
-10
Br, [G]

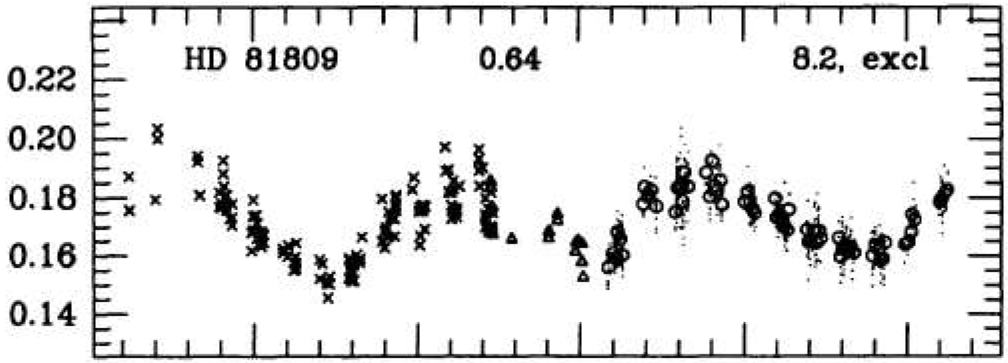
HK survey, Ca II H & K lines (Baliunas et al. 1995)

Rotatioin rates from SIMBAD

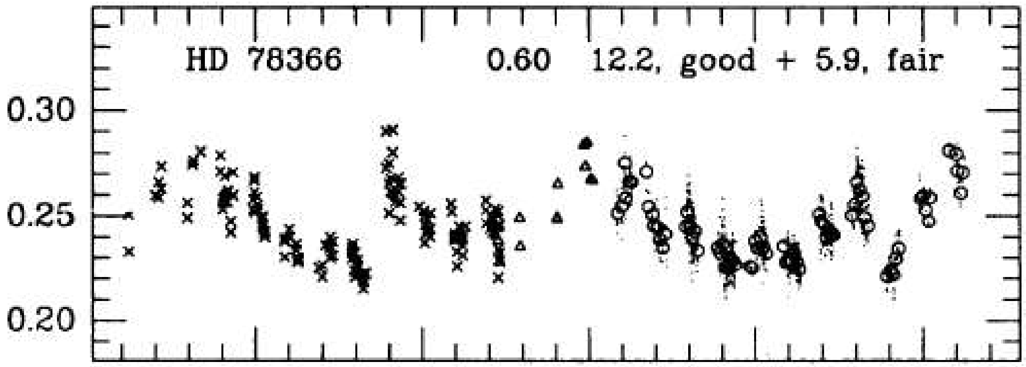
Sun **G2 V** **2km/s**



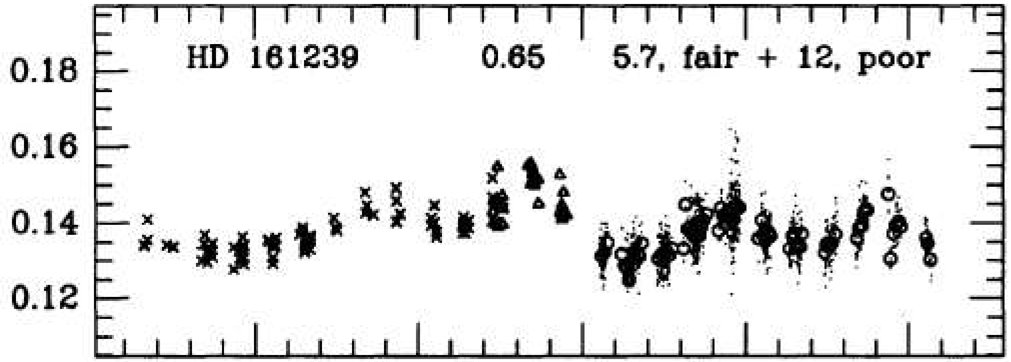
G2 V **Vsini~5 km/s**



G0 V **Vsini~5.9 km/s**



G6 V **Vsini~5. km/s**



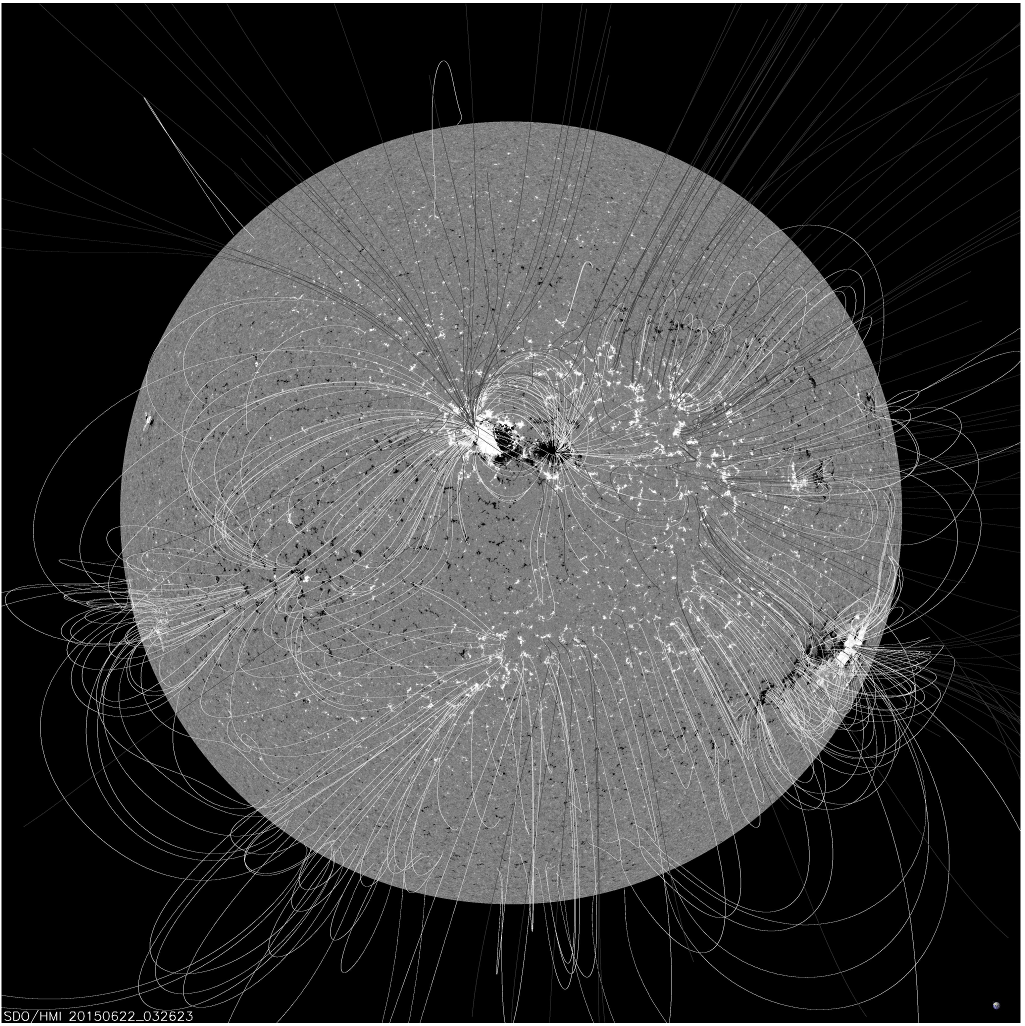
1970 **1975** **1980** **1985** **1990**

1970 **1975** **1980** **1985** **1990**

The Sun

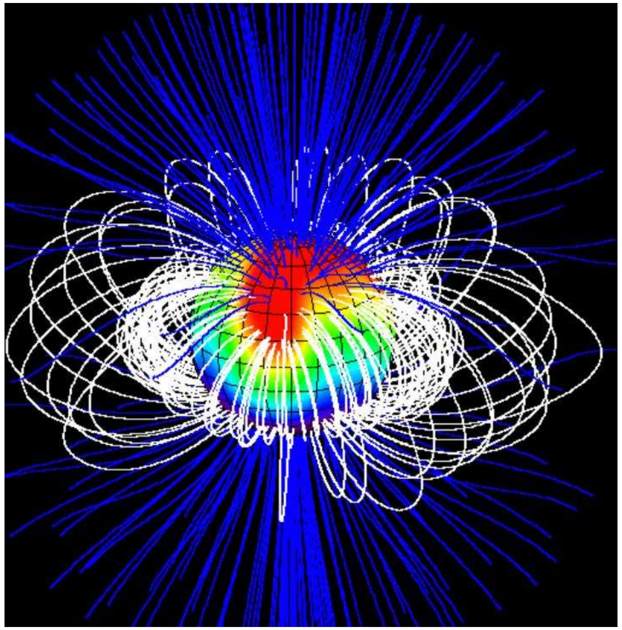
M-dwarf

SDO/HMI,
~2G



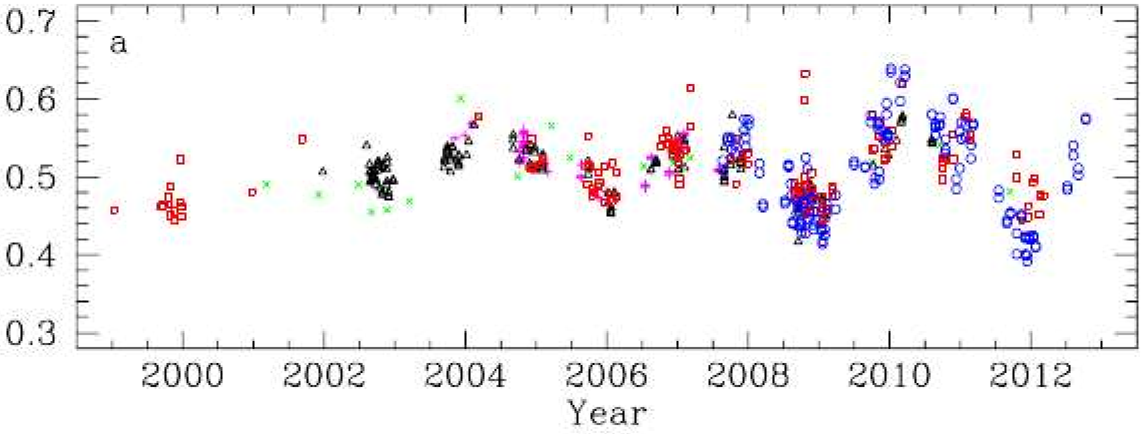
V374 Peg, 0.3Ms
rotating with period
of a half day,
~2kG

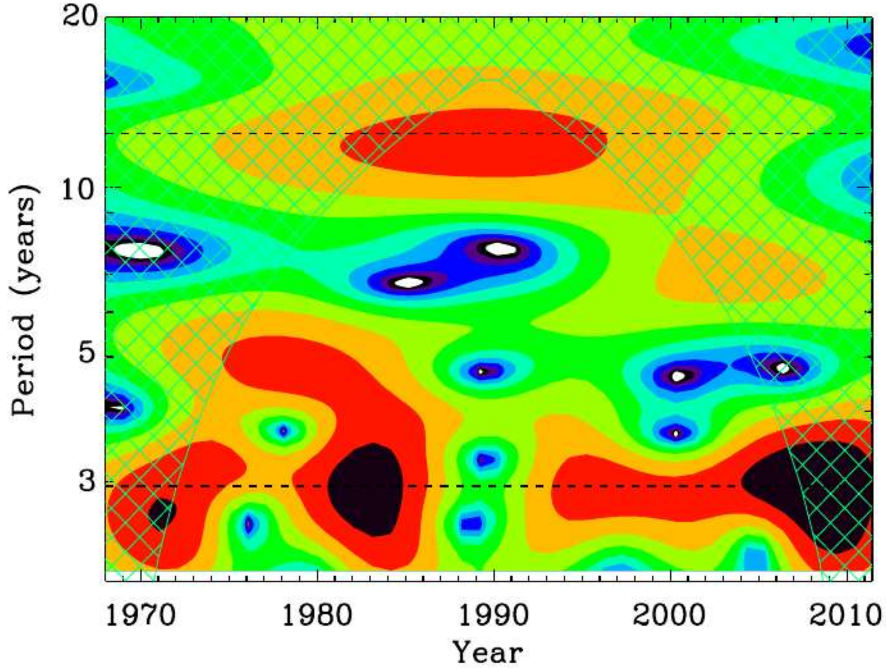
(Donati et al 2009, ℓ<13)



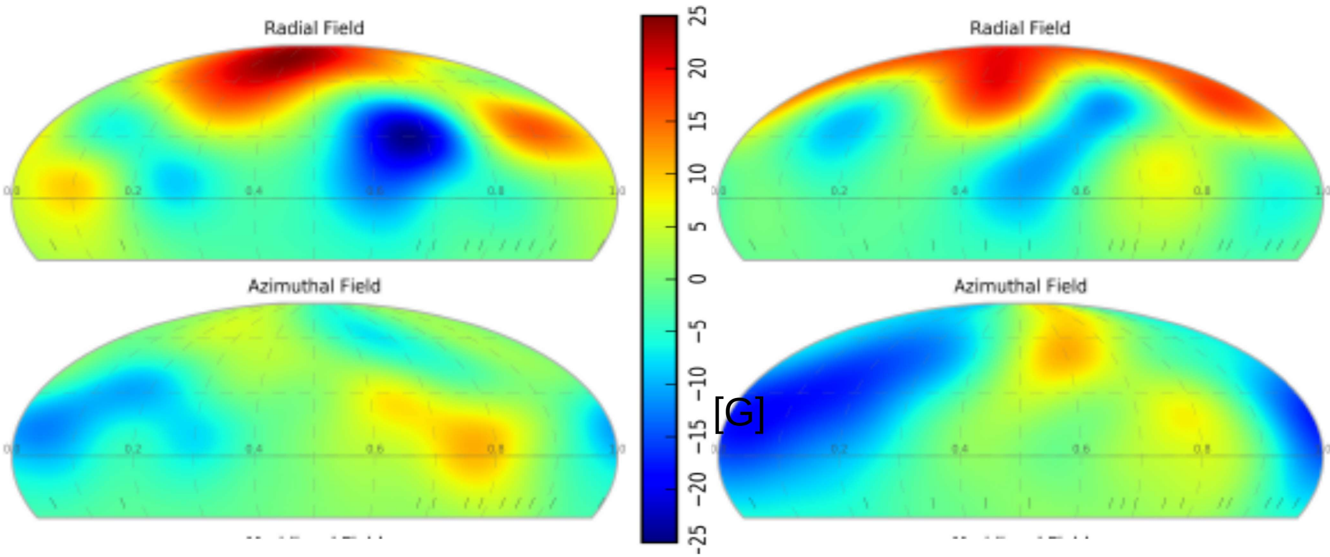
Solar “twin”, ε-Eridani (K2, 0.85Ms,P_{rot}=11d)

Metcalfe et al. 2013



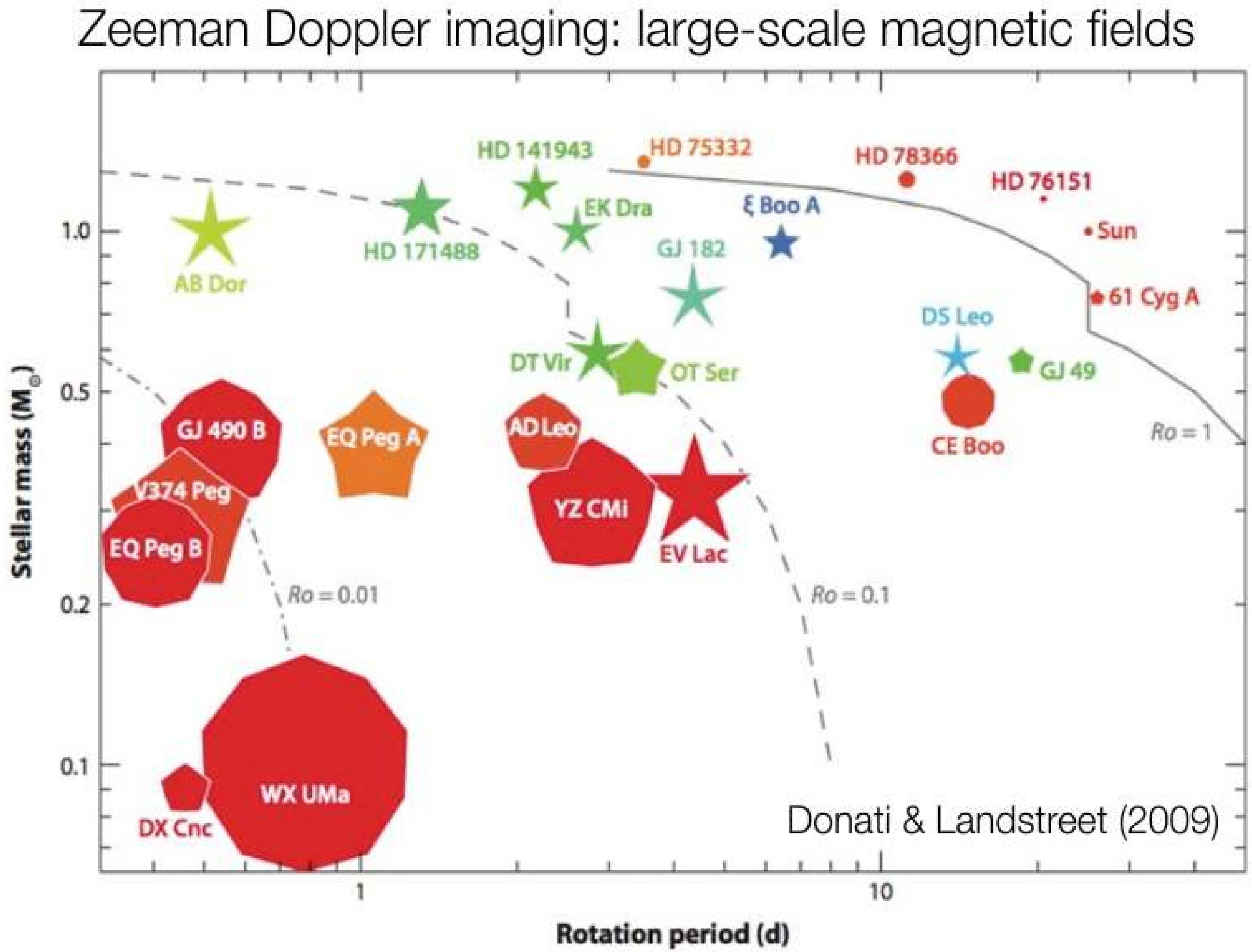


Jeffers et al. 2017



Magnetic field topology is very diverse...

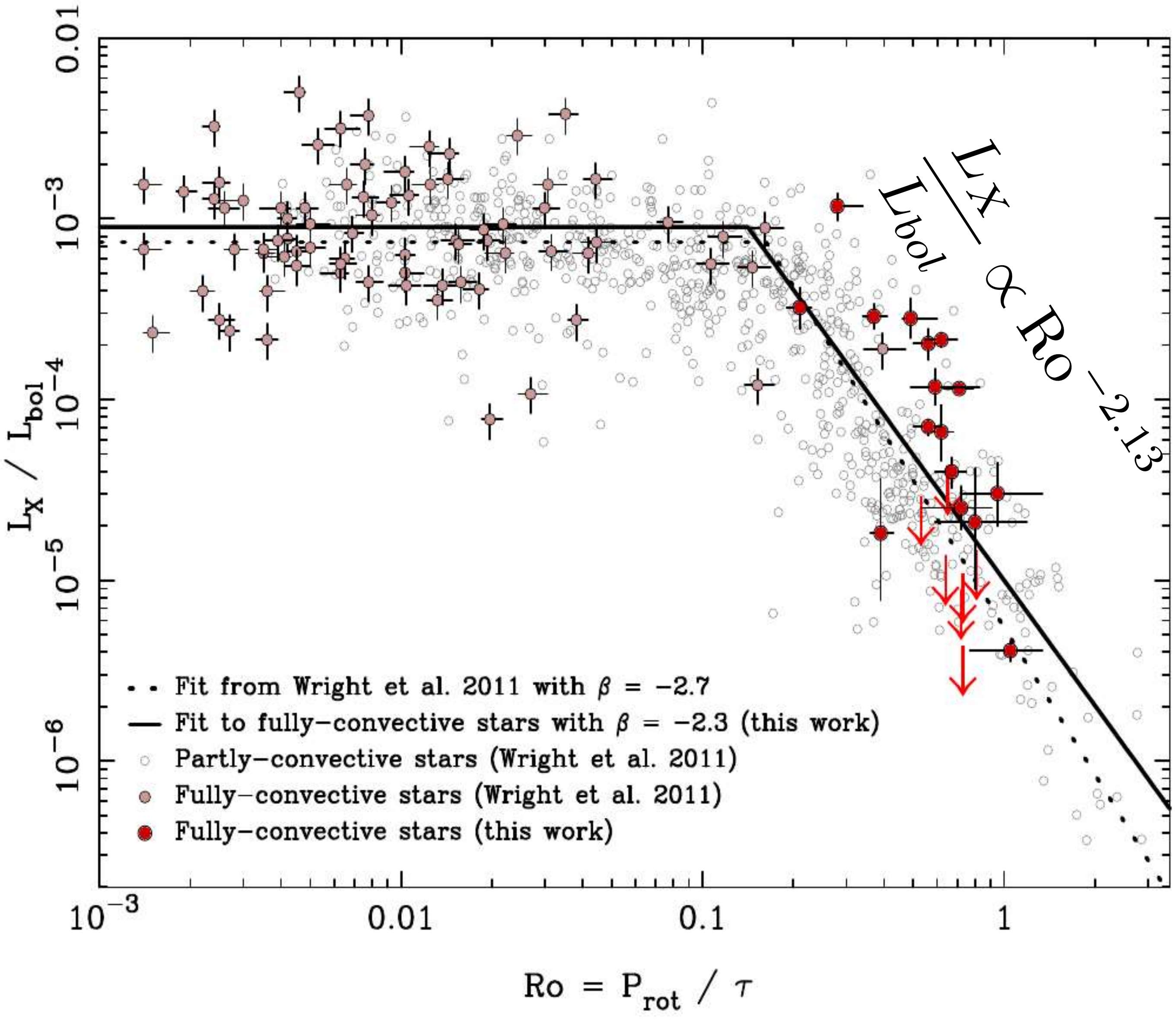
Credit to Vidotto (2018)



- Size: magnetic energy
- Colour: purely toroidal (blue) or poloidal (red) fields.
- Shape: purely axisymmetric (decagon) or non-axisymymmetric (star).

Magnetic topology depends on stellar mass and rotation. The strongest MF is observed on M-dwarfs.

Stellar activity magnitude, X-ray



Wright et al (2018):

Do we need tachocline?

Is there a common dynamo process among fully and partially convective stars?

Resume

- The Sun’s type magnetic activity is organized on the large-scales in time and space
- Global rotation defines
  - Topology of magnetic field
  - The overall strength of magnetic activity
- Convection (stellar mass) defines
  - Amount of turbulent energy for the dynamo
  - Spots formation and etc

Formal scheme

$$\mathbf{v} = \langle \mathbf{U} \rangle + \mathbf{u}, \quad \mathbf{B} = \langle \mathbf{B} \rangle + \mathbf{b}, \quad s = \langle S \rangle + s', \quad \ldots$$

→ MHD Equations

$\langle \cdots \rangle$ - Ensemble averaging

One-fluid ideal magnetohydrodynamics

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}),$$

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \left(\mathbf{v} \cdot \nabla \right) \mathbf{v} = \nabla p + \rho \mathbf{g} + \frac{1}{4\pi} \left(\nabla \times \mathbf{B} \right) \times \mathbf{B}$$

$$\rho T \left[\frac{\partial s}{\partial t} + \left(\mathbf{v} \cdot \nabla \right) s \right] = \boldsymbol{\nabla} \cdot \left(\mathbf{F}^c + \mathbf{F}^r \right) + \varepsilon^s + \ldots$$

Ensemble averaging

- For an ensemble averaging we use the set of the identical systems (the member of the given ensemble) where the measurements are carried on. This concept is based on the probability:

$$\langle \mathbf{F}(t_1) \rangle = \int P_t(\mathbf{F}) \, \mathbf{F}_t d\mathbf{F}_t$$

Examples:

- In the laboratory, (say in Perm where they have a dynamo machine) in studying the magnetic field in turbulent channel.
- The theoretical study of the turbulent flows.
- For the Sun we might assume that each 22 cycle is a separate member of an ensemble.

Advantage:

- Reynods averaging rule
- No scale-separation asumptions

Disadvantage:

- Hard to perform in observations

Reynolds averaging rules

$$\langle \mathbf{F}_1 + \mathbf{F}_2 \rangle = \langle \mathbf{F}_1 \rangle + \langle \mathbf{F}_2 \rangle$$

$$\langle \langle \mathbf{F}_1 \rangle \mathbf{F}_2 \rangle = \langle \mathbf{F}_1 \rangle \langle \mathbf{F}_2 \rangle$$

$$\langle a\mathbf{F}_1 \rangle = a \langle \mathbf{F}_1 \rangle$$ If a=const

They are valid for ergodic ensemble systems

Temporal averaging:

$$\langle \mathbf{B}(\mathbf{x},t)\rangle_t = \frac{1}{T_a}\int (\mathbf{B}(\mathbf{x},t) + \mathbf{b}(\mathbf{x},t))\, dt$$

How to prescribe T_a ?

- The practical approach would be to check the condition:

$$\frac{1}{T_a}\int \mathbf{b}(\mathbf{x},t)\, dt = 0$$

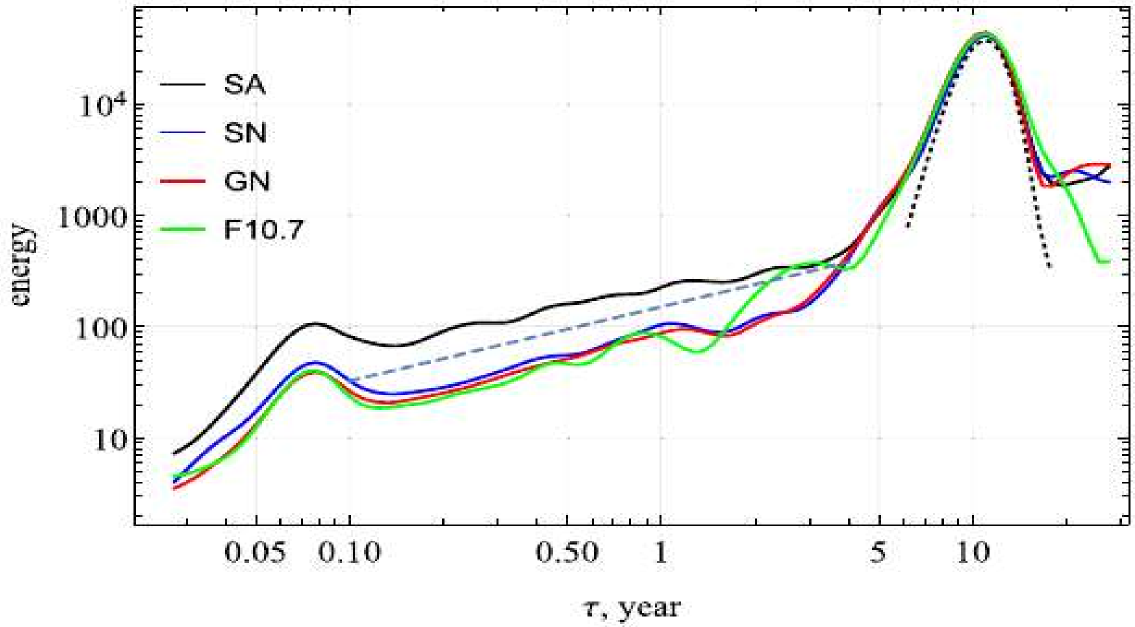
- Scale-separation assumption is needed if : $\partial_t \mathbf{b} \sim \frac{\mathbf{b}}{\tau_c}$

Then $T_a \gg \tau_c$

For the Sun, the largest convective cells (occupying the most depths' of CZ) have $\tau_c \approx 1$ - 3 months (typical life-time of the large AR), then $T_a > 1$ Yr.

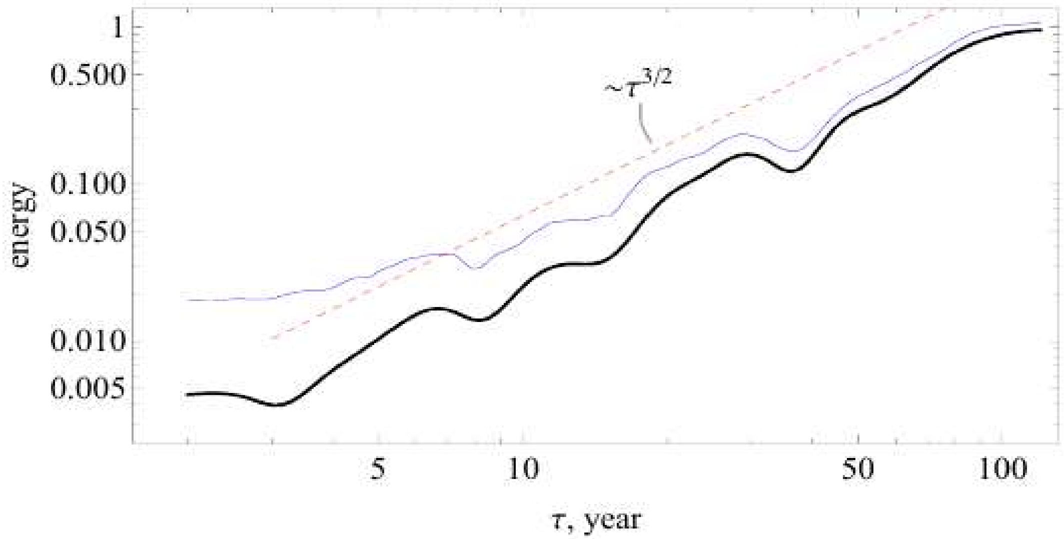
Does it work for the Sun?

Sun



Frick et al 2020

V833 Tau



Stepanov et al 2020

Spatial averaging:

$$\langle \mathbf{B}(\mathbf{x},t)\rangle_x = \frac{1}{L_a^3}\int \left(\mathbf{B}(\mathbf{x},t)+\mathbf{b}(\mathbf{x},t)\right)d\mathbf{x}$$

How to prescribe L_a ?

- Again, **the scale-separation assumption is needed,** if :

$$\partial_{\mathbf{x}}\mathbf{b} \sim \frac{\mathbf{b}}{\ell_c}, \qquad \text{and} \qquad \partial_{\mathbf{x}}\mathbf{B} \sim \frac{\mathbf{B}}{L_a},$$

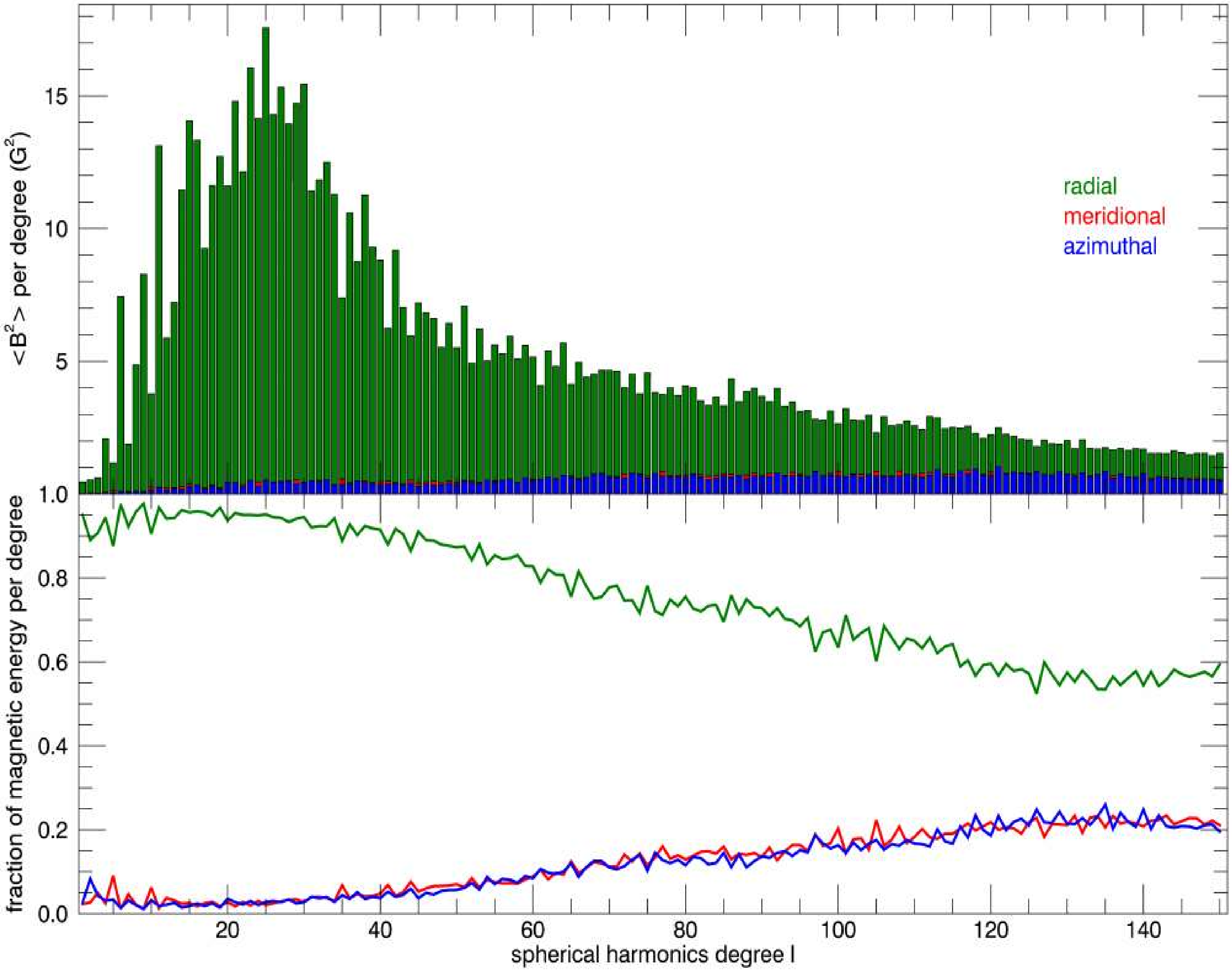
then for $L_a \gg \ell_c$ we can use the spatial averaging

- For the Sun, the largest convective cells (occupying the balk of CZ) have $\ell_c \approx 0.2\ R_\odot \sim 12^o$ - the typical size of the large AR.

- In athimuth the axisymmetric magnetic field satisfies $L_a \gg \ell_c$ while in latitudinal and radial direction it is not

Spatial scale distribution for the magnetic Sun

Vidotto 2016



Formal scheme

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$$\rho T \left[\frac{\partial s}{\partial t} + \left(\mathbf{v} \cdot \nabla\right) s \right] = \boldsymbol{\nabla} \cdot \left(\mathbf{F}^c + \mathbf{F}^r\right) + \varepsilon^s + \ldots$$

Mean-field equations

$$\partial_{\mathrm{t}} \langle \mathbf{B} \rangle = \boldsymbol{\nabla} \times (\boldsymbol{\mathcal{E}} + \langle \mathbf{U} \rangle \times \langle \mathbf{B} \rangle),$$

$$\frac{\partial}{\partial \mathrm{t}} \overline{\rho} \mathrm{r}^2 \sin^2 \theta \Omega \;\; = - \boldsymbol{\nabla} \cdot \left(r \sin\theta \left(\overline{\rho} \hat{\mathbf{T}}_{\phi} + r \overline{\rho} \sin\theta \Omega \overline{\mathbf{U}}^{\mathbf{m}} \right) \right)$$

$$+ \quad \boldsymbol{\nabla} \cdot \left(r \sin\theta \frac{\langle \mathbf{B} \rangle \langle B_{\phi} \rangle}{4\pi} \right)$$

$$\overline{\rho}\overline{T} \left(\frac{\partial \langle \mathrm{s} \rangle}{\partial t} + (\langle \mathbf{U} \rangle \cdot \boldsymbol{\nabla}) \langle \mathrm{s} \rangle \right) = - \boldsymbol{\nabla} \cdot (\mathbf{F}^{c} + \mathbf{F}^{r}) - \hat{T}_{ij} \frac{\partial \langle U \rangle_i}{\partial r_j} - \boldsymbol{\mathcal{E}} \cdot (\nabla \times \langle \mathbf{B} \rangle) ,$$

$$\boldsymbol{\mathcal{E}} = \langle \mathbf{u} \times \mathbf{b} \rangle$$

$$\hat{T}_{ij} = \left(\langle u_i u_j \rangle - \frac{1}{4\pi\overline{\rho}} \left(\langle b_i b_j \rangle - \frac{1}{2} \delta_{ij} \langle \mathbf{b}^2 \rangle \right) \right),$$

$$\mathrm{F}_{\mathrm{i}}^{\mathrm{conv}} = -\mathrm{c_p} \overline{\rho} \overline{\mathrm{T}} \kappa_{\mathrm{ij}} \nabla_{\mathrm{j}} \langle \mathrm{s} \rangle, \;\; \kappa_{ij} = \langle u_i u_j \rangle$$

} Effect of turbulent flows and magnetic field

Equations driving the turbulence:
Model 1: shochastically driving turbulence

$$\frac{\partial \mathbf{b}}{\partial t} = \nabla \times (\mathbf{u} \times \langle \mathbf{B} \rangle + \langle \mathbf{U} \rangle \times \mathbf{b}) + \eta \nabla^2 \mathbf{b} + \nabla \times (\mathbf{u} \times \mathbf{b} - \boldsymbol{\mathcal{E}}) + \mathfrak{G},$$

$$\begin{aligned} \rho \frac{\partial u_i}{\partial t} + 2\rho \left(\mathbf{\Omega} \times \mathbf{u} \right)_i &= -\nabla_i \left(p + \frac{(\mathbf{b} \cdot \langle \mathbf{B} \rangle)}{4\pi} \right) + \nu \Delta \rho u_i \\ &+ \frac{1}{4\pi} \nabla_j \left(\langle B_j \rangle b_i + \langle B_i \rangle b_j \right) + \nabla_j \left(\rho T_{i,j} - \rho \hat{T}_{i,j} \right) \\ &- \nabla_j \left(\rho \langle U_j \rangle u_i + \rho \langle U_i \rangle u_j \right) + f_i + \mathfrak{F}_i, \end{aligned}$$

Here, G and F are random forces driving turbulence, their statistics is given. In other words, the properties of the background turbulence are given

$$\frac{\partial \mathbf{b}^{(0)}}{\partial t} = \eta \nabla^2 \mathbf{b}^{(0)} + \nabla \times \left(\mathbf{u}^{(0)} \times \mathbf{b}^{(0)} \right) + \mathfrak{G},$$

$$\rho \frac{\partial u_i^{(0)}}{\partial t} = -\nabla_i p + \nu \Delta \rho u_i^{(0)} + \nabla_j \left(\rho T_{i,j}^{(0)} \right) + f_i + \mathfrak{F}_i,$$

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Governing parameters:

∂_t **b**, $\eta\Delta$ **b** versus $\nabla\times$ ($\mathbf{u}\times$ **b**) → (St=uτ/ℓ , Rm=uℓ/η)

∂_t **u**, $\nu\Delta$ **u** versus $\nabla\cdot$ $\mathbf{u}\otimes$ **u** → (St=uτ/ℓ , Re=uℓ/ν)

Basic parametres to quantify nonlinear effects of large-scale field:

$$\Omega^* = 2\Omega\tau_c = Ro^{-1} \qquad \beta = |\langle B \rangle| / \sqrt{4\pi\bar{\rho}\,\langle u^2 \rangle}$$

Equations driving the turbulence: Model 2: quasiadiabatic stellar convection

$$\begin{aligned}
\rho\frac{\partial u_i}{\partial t} + 2\rho\left(\mathbf{\Omega}\times\mathbf{u}\right)_i &= -\nabla_i\left(p + \frac{(\mathbf{b}\cdot\langle\mathbf{B}\rangle)}{4\pi}\right) + \nu\Delta\rho u_i \\
&+ \frac{1}{4\pi}\nabla_j\left(\langle B_j\rangle b_i + \langle B_i\rangle b_j\right) + \nabla_j\left(\rho T_{i,j} - \rho\hat{T}_{i,j}\right) \\
&- \nabla_j\left(\rho\langle U_j\rangle u_i + \rho\langle U_i\rangle u_j\right) + \rho g_i,
\end{aligned}$$

- Introduce the reference state: $\boldsymbol{\nabla}\bar{P} = \mathbf{g}\bar{\rho}$
- Consider equation of state for density variations:

$$\frac{\rho'}{\bar{\rho}} = \frac{p'}{\gamma\bar{P}} - \frac{s'}{c_p},$$

$$\frac{\boldsymbol{\nabla}\langle s\rangle}{c_p} = \frac{\boldsymbol{\nabla}\bar{P}}{\gamma\bar{P}} - \frac{\boldsymbol{\nabla}\bar{\rho}}{\bar{\rho}}$$

- Employ the Boussinesque approximation (neglect density variations except buoancy forces)

Model 2: quasiadiabatic stellar convection

$$\bar{\rho}\frac{\partial u_i}{\partial t} + 2\bar{\rho}\left(\mathbf{\Omega}\times\mathbf{u}\right)_i = -\bar{\rho}\nabla_i\left(\frac{p'}{\bar{\rho}}\right) - \nabla_i\left(\frac{(\mathbf{b}\cdot\langle\mathbf{B}\rangle)}{4\pi}\right) + \nu\Delta\bar{\rho}u_i$$

$$+ \frac{1}{4\pi}\nabla_j\left(\langle B_j\rangle b_i + \langle B_i\rangle b_j\right) + \nabla_j\left(\rho T_{i,j} - \rho\hat{T}_{i,j}\right)$$

$$- \nabla_j\left(\bar{\rho}\langle U_j\rangle u_i + \bar{\rho}\langle U_i\rangle u_j\right) - \bar{\rho}\frac{g_i}{c_p}s',$$

$$\overline{\rho}\overline{T}\left(\frac{\partial s'}{\partial t} + (\langle\mathbf{U}\rangle\cdot\boldsymbol{\nabla})s'\right) = -\overline{\rho}\overline{T}\left(\mathbf{u}\cdot\boldsymbol{\nabla}\langle s\rangle\right) + \boldsymbol{\nabla}\cdot\kappa\bar{\rho}\bar{T}\boldsymbol{\nabla}s'$$

$$\frac{\partial\mathbf{b}}{\partial t} = \nabla\times\left(\mathbf{u}\times\langle\mathbf{B}\rangle + \langle\mathbf{U}\rangle\times\mathbf{b}\right) + \eta\nabla^2\mathbf{b} + \nabla\times\left(\mathbf{u}\times\mathbf{b} - \boldsymbol{\mathcal{E}}\right) + \mathfrak{G},$$

Governing parameters:

∂_t **b**, $\eta\Delta$ **b** versus $\nabla\times$ (**u**$\times$ **b**) → (St=uτ/ℓ , Rm=uℓ/η)

∂_t **u**, $\nu\Delta$ **u** versus $\nabla\cdot$ **u**$\otimes$ **u** → (St=uτ/ℓ , Re=uℓ/ν)

∂_t s' versus κ Δs' → Pe=uℓ/κ

$\nu\Delta$ **u** versus **g** s'/c_p (where s'~u $\ell\nabla$ <s>/κ)→ Ra= $\dfrac{g\ell^3\left|\nabla\langle s\rangle\right|}{c_p\kappa\nu}$

$$\Omega^* = 2\Omega\tau_c = Ro^{-1} \qquad \beta = |\langle B\rangle|/\sqrt{4\pi\bar{\rho}\,\langle u^2\rangle}$$

Basic parameters of the solar convection zone plasma

	ℓ, [cm]	τ, [s]	u, [cm/s]	κ, [cm²/s]	ν, [cm²/s]	η,[cm²/s]	Re	Rm	Ra
r≈2/3R	$5\ 10^9$	$3\ 10^6$	$2\ 10^3$	10^3	10^{-1}	10^{-4}	10^{14}	10^{17}	10^{21}
R≈0.99R	$3\ 10^7$	10^3	10^5	10	10^{-2}	10^{-9}	10^{14}	10^{22}	10^{20}

Here, we use $\frac{|\nabla \langle s \rangle|}{c_p} \sim 10^{-8} - 10^{-6}$

Stellar convection zones: $\eta < \nu < \kappa$