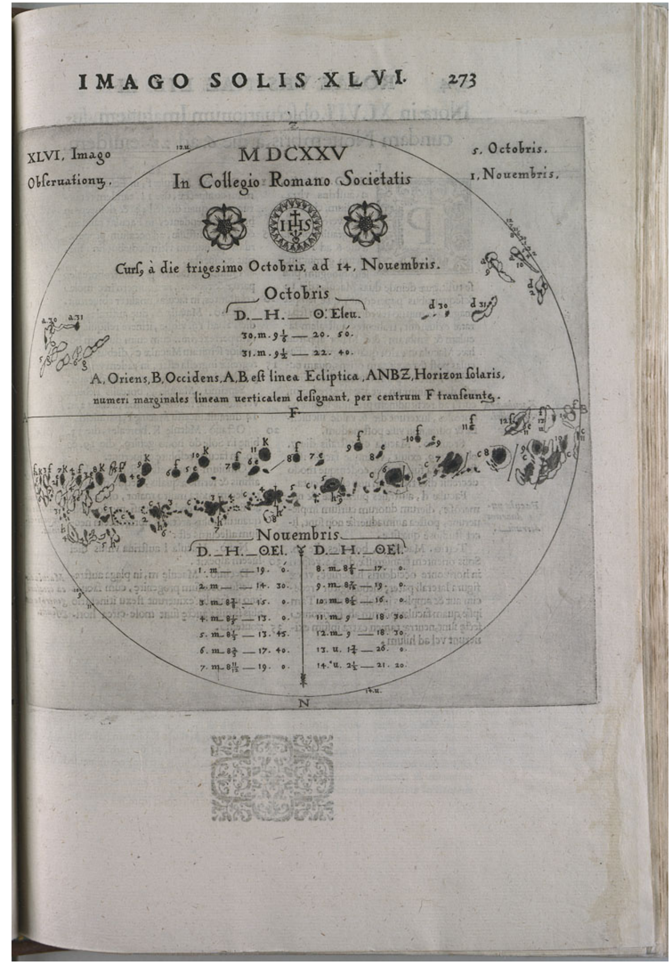
Differential rotation and turbulent angular momentum transport

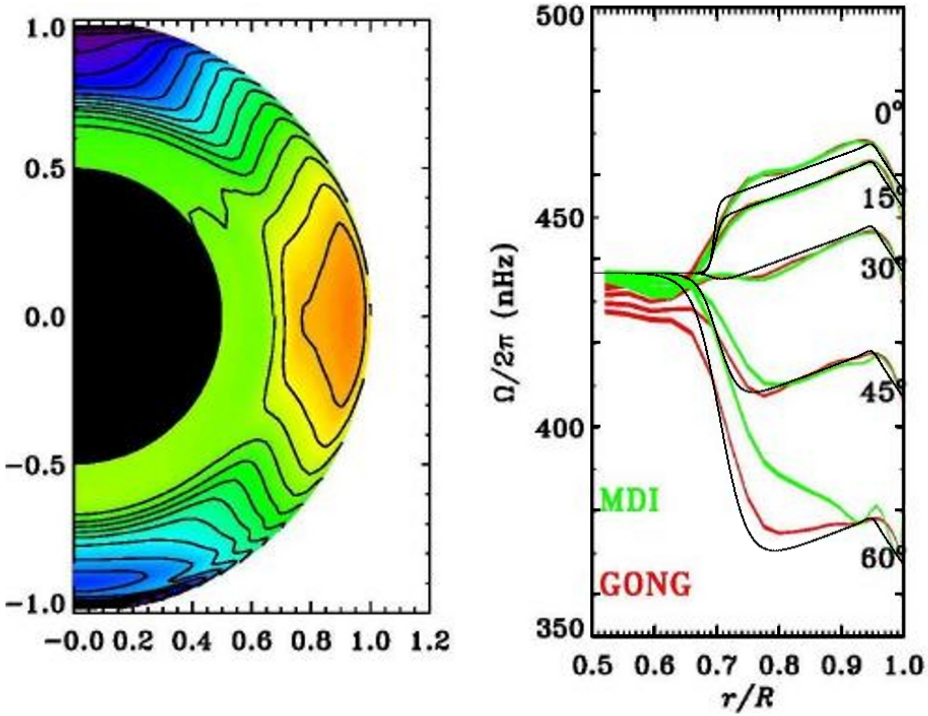
Topics:

1. The solar and stellar differential rotation and cirulation
2. Anisotropic heat flux
3. Turbulent angular momentum transport
4. Results from the mean-field models

Discovering by C. Schineir (1630)

- J. Fabricius "Narration on Spots Observed on the Sun and their Apparent Rotation with the Sun".
- Christopher Scheiner ("Rosa Ursine sive solis", book 4, part 2, 1630) was the first to measure the equatorial rotation rate of the Sun.





- Latitudinal shear in the solar convection zone. Radial shear near the bottom and near the top.
- In equatorial region the angular velocity increase outwards.
- Almost steady. Though, the short-term 11th year variations about 5m/s and and the long-term variations ~50-100m/s.
- There are indications for strong DR during Mounder minimum

Meridional circulation (see, Chen & Zhao 2024)

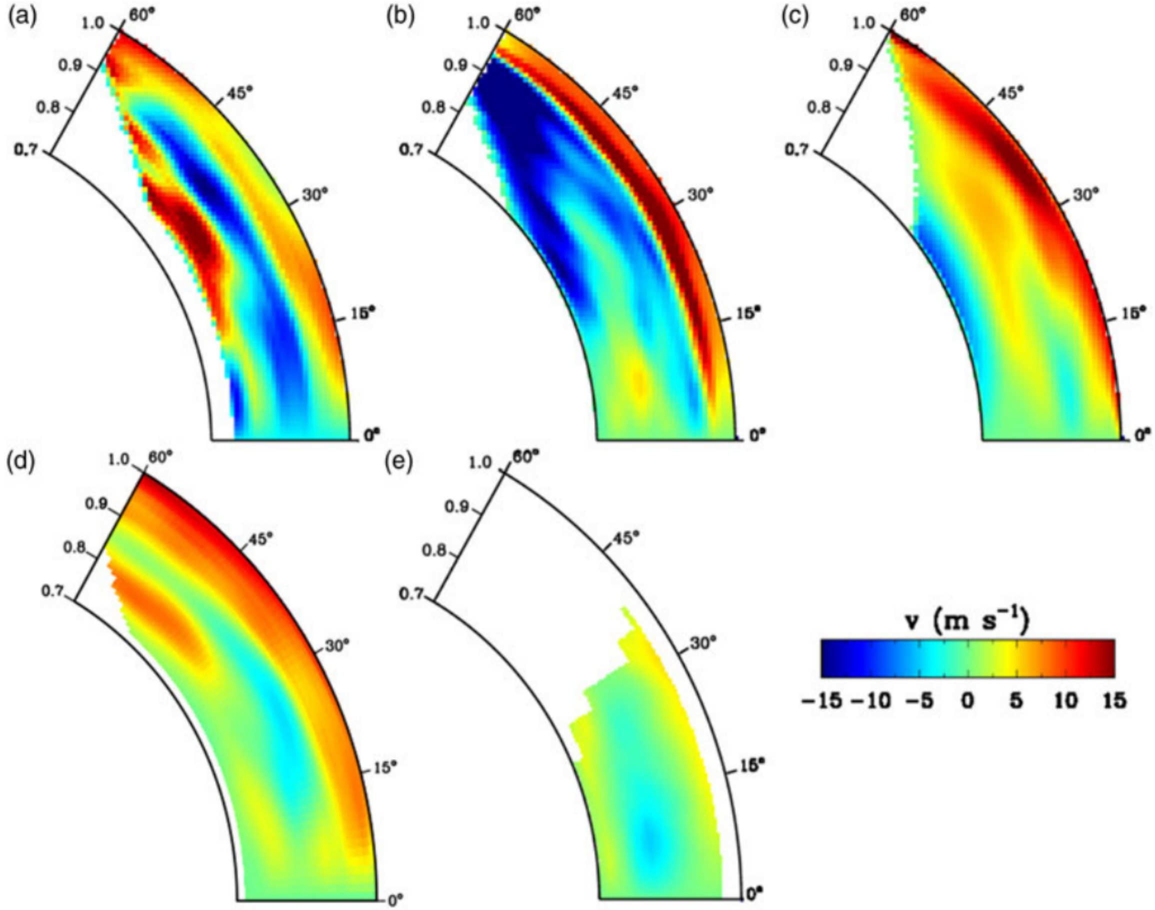


Figure 1. Comparison of hemispherically symmetrized meridional circulation for selected results by time–distance method, from (a) Zhao *et al.* (2013), (b) Kholikov *et al.* (2014); Jackiewicz *et al.* (2015), (c) Rajaguru & Antia (2015), (d) Chen & Zhao (2017), and (e) Lin & Chou (2018). Adapted from Figure 1.8 of Chen (2019).

Conservation of angular momentum

The mean-field equation:

$$\frac{\partial}{\partial t}\bar{\rho} r^2 \sin^2\theta\Omega \;=\; -\boldsymbol{\nabla}\cdot\left(r\sin\theta\left(\bar{\rho}\hat{\mathbf{T}}_\phi + \bar{\rho}\, r\, \sin\theta\Omega\langle \boldsymbol{U}\rangle - \frac{\langle \boldsymbol{B}\rangle\langle B_\phi\rangle}{4\pi}\right)\right)$$

Effects of the turbulent flows and magnetic fields:

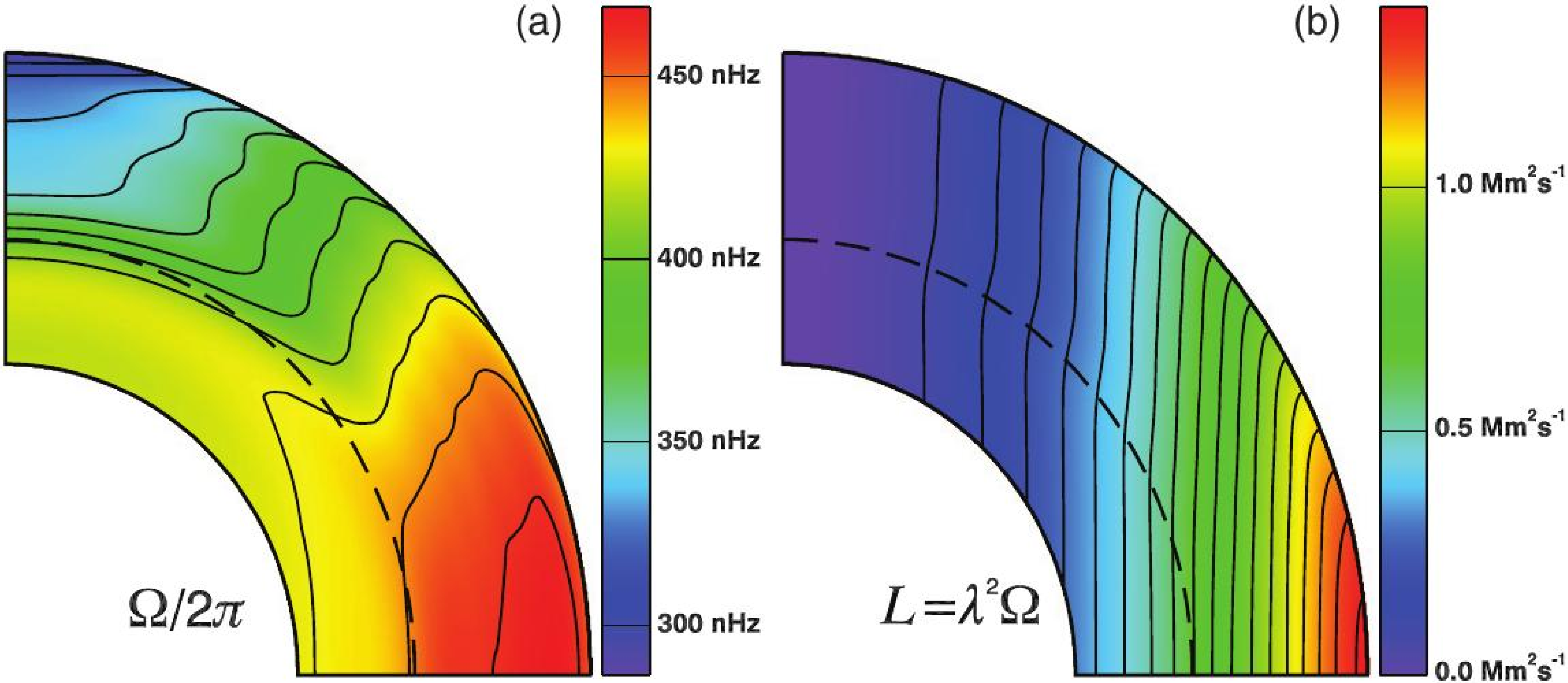
$$\hat{T}_{\rm ij} \;=\; \langle u_i u_j\rangle - \frac{1}{4\pi\bar{\rho}}(\langle b_i b_j\rangle - \delta_{\rm ij}\langle b^2\rangle), \hat{\mathbf{T}}_\phi = \hat{T}_{\rm ij} e^j_\varphi$$

Let us consider a steady state with $\langle \boldsymbol{B}\rangle = 0$, then

$$\bar{\rho}(\langle \boldsymbol{U}\rangle \cdot \boldsymbol{\nabla})\mathcal{L} = -\boldsymbol{\nabla}\cdot(r\sin\theta(\bar{\rho}\hat{\mathbf{T}}_\phi))$$

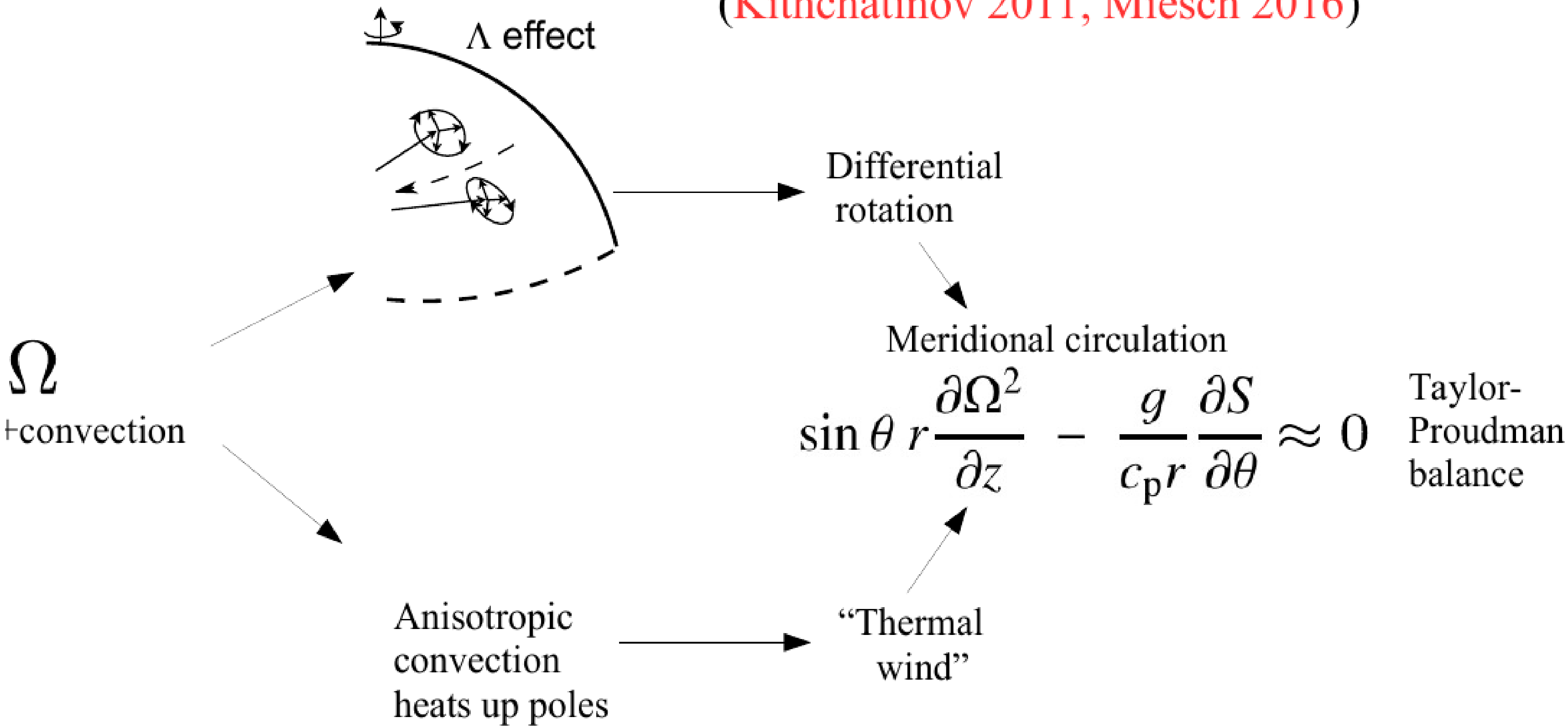
where, $\mathcal{L} = r^2\sin^2\Omega$. So, if $\hat{T}_{\rm ij} = 0$, then $\bar{\rho}(\langle \boldsymbol{U}\rangle \cdot \boldsymbol{\nabla})\mathcal{L} = 0$, i.e., in absence of turbulent stresses MC streamlines = angular momentum contours. This is not the case of the Sun!

Angular momentum profile



General scheme

(Kithchatinov 2011, Miesch 2016)



Anisotropic heat transport

The large Reynolds number limit. Consider the forced turbulence

$$\frac{\boldsymbol{u}}{\boldsymbol{\tau_c}} \approx \boldsymbol{2u} \times \boldsymbol{\Omega} + \frac{\boldsymbol{u}^{(0)}}{\boldsymbol{\tau_c}} \cdots$$

Let us divide the flow into sum along and perpendecular to the rotation axis:

$$\boldsymbol{u} = \boldsymbol{u}_{\perp} + \boldsymbol{u}_{\|}, \boldsymbol{u}_{\perp} = \boldsymbol{u} - \frac{\boldsymbol{\Omega}}{\boldsymbol{\Omega^2}}(\boldsymbol{u} \cdot \boldsymbol{\Omega}); \boldsymbol{u}_{\|} = \frac{\boldsymbol{\Omega}}{\boldsymbol{\Omega^2}}(\boldsymbol{u} \cdot \boldsymbol{\Omega})$$

Assume that in the background turbulence (without global rotation) is isotropic, $\langle u_{\perp}^{(0)2} \rangle = \langle u_{\|}^{(0)2} \rangle$. Then

$$(\delta_{\rm ij} - 2\tau_c \varepsilon_{\rm ijn} \Omega_n) u_{\perp j} \;=\; u_{\perp i}^{(0)}; u_{\| i} = u_{\| i}^{(0)}$$

Consider intensity of turbulent flows, $\langle u_{\perp}^2 \rangle$ and $\langle u_{\|}^2 \rangle$ under effect of the global rotation:

$$(1 + 4\tau_c^2 \Omega^2) \langle u_{\perp}^2 \rangle \;=\; \langle u_{\perp}^{(0)2} \rangle; \langle u_{\|}^2 \rangle = \langle u_{\|}^{(0)2} \rangle$$

Here we again employ identity $\varepsilon_{\rm ijn} u_{\perp j} \Omega_n \varepsilon_{\rm ipm} u_{\perp p} \Omega_m = (\delta_{\rm jp}\delta_{\rm nm} - \delta_{\rm jm}\delta_{\rm np}) u_{\perp j} \Omega_n u_{\perp p} \Omega_m$

The mean-field heat-transport equation:

$$\bar{\rho}\bar{T}\left(\frac{\partial\left\langle s\right\rangle}{\partial t}+\left(\left\langle \boldsymbol{U}\right\rangle\cdot\boldsymbol{\nabla}\right)\left\langle s\right\rangle\right) = -\boldsymbol{\nabla}\cdot\left(\mathbf{F}^{c}+\mathbf{F}^{\mathrm{rad}}\right)-\hat{T}_{\mathrm{ij}}\frac{\partial\left\langle U_{i}\right\rangle}{\partial r_{j}}-\boldsymbol{\mathcal{E}}\cdot\boldsymbol{\nabla}\times\left\langle \boldsymbol{B}\right\rangle$$

The effect of turbulent convection:

$$F_{i}^{c} = -c_{p}\bar{\rho}\bar{T}\kappa_{\mathrm{ij}}\nabla_{j}\left\langle s\right\rangle ,\kappa_{\mathrm{ij}}=\left\langle u_{i}u_{j}\right\rangle$$

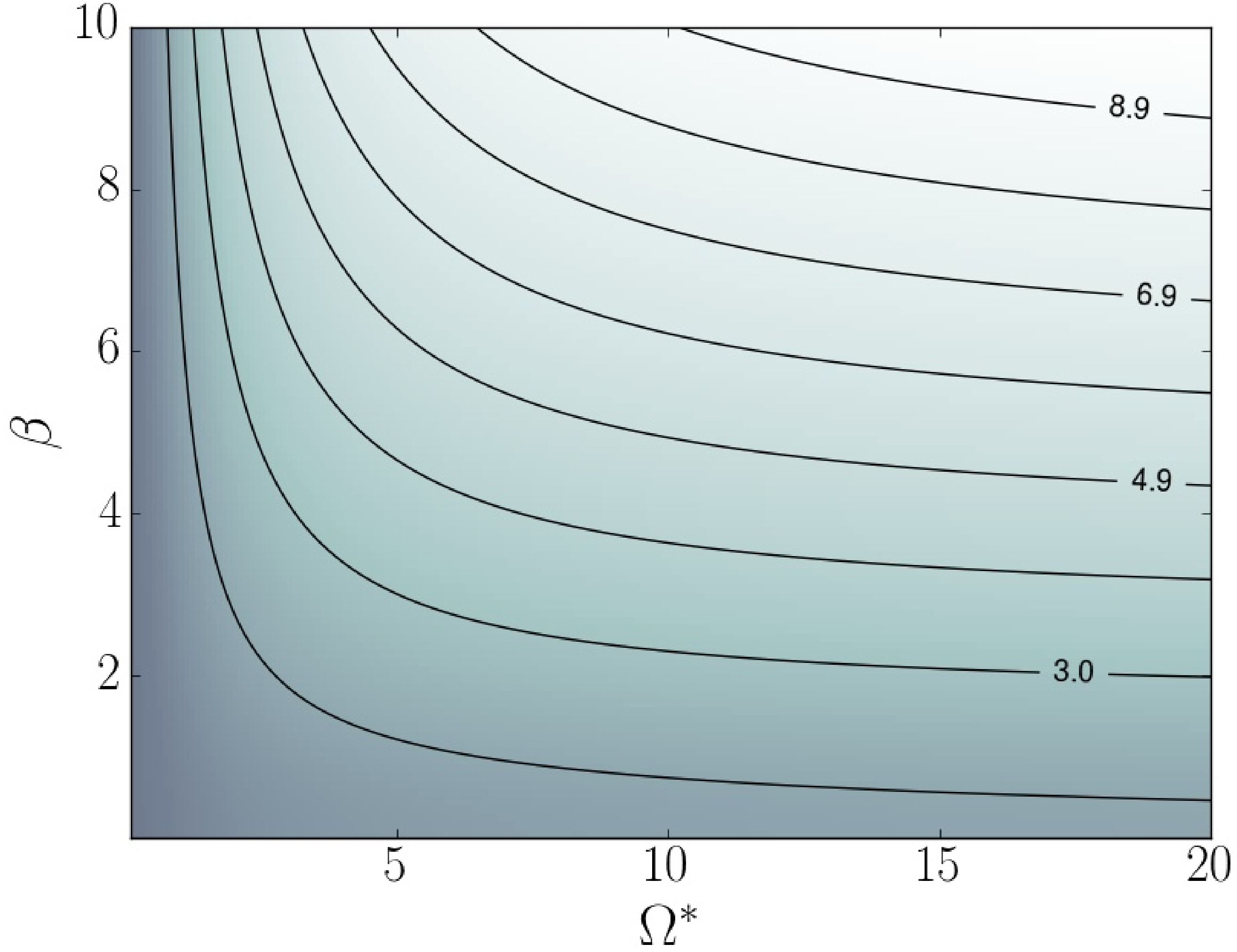
Here, I show the results of Pipin (2004):

$$\kappa_{\mathrm{ij}}=\kappa_{T}\left(\psi_{\kappa}^{(I)}(\beta)\psi(\Omega^{*})\delta_{\mathrm{ij}}+\psi_{\kappa}^{(A)}(\beta)\psi_{||}(\Omega^{*})\frac{\Omega_{i}\Omega_{j}}{\Omega^{2}}\right)$$

$\kappa_{\perp}$ perpendecular to $\boldsymbol{\Omega}$ $\quad + \quad$ $\kappa_{||}$ is along $\boldsymbol{\Omega}$

Here, $\beta=\left|\left\langle \boldsymbol{B}\right\rangle \right|/\sqrt{4\pi\bar{\rho}\left\langle u^{(0)2}\right\rangle }$, and $\Omega^{*}=2\tau_{c}\Omega$ is the Coriolis number.

Anisotropy $\kappa_{\|}/\kappa_{\perp}$, for large β and Ω^*



Turbulent angular momentu transport

We have to solve:

$$\begin{aligned}
\partial_t \boldsymbol{b} &= \boldsymbol{\nabla} \times (\boldsymbol{u} \times \langle \boldsymbol{B} \rangle + \langle \boldsymbol{U} \rangle \times \boldsymbol{b}) + \eta \nabla^2 \boldsymbol{b} + \nabla \times (\boldsymbol{u} \times \boldsymbol{b} - \mathcal{E}) + \mathfrak{G} \\
\bar\rho \partial_t u_i &= 2\bar\rho (\boldsymbol{u} \times \boldsymbol{\Omega})_i - \nabla_i \left(p + \frac{(\boldsymbol{b} \cdot \langle \boldsymbol{B} \rangle)}{8\pi} \right) + \nu \Delta \bar\rho u_i + f_i + \mathfrak{F}_i \\
&- \nabla_j (\bar\rho u_i \langle U_j \rangle + u_j \langle U_i \rangle) + \nabla_j (\bar\rho T_{\rm ij} - \bar\rho \hat{T}_{\rm ij}) \\
&+ \frac{1}{4\pi} \nabla_j (b_i \langle B_j \rangle + b_j \langle B_i \rangle)
\end{aligned}$$

To determine

$$\hat{T}_{\rm ij} = \langle u_i u_j \rangle - \frac{1}{4\pi\bar\rho} (\langle b_i b_j \rangle - \delta_{\rm ij} \langle b^2 \rangle)$$

This can be done using the FOSA (Kitchatinov, Ruediger & Pipin, 1993,1994,1996,2004) or using the τ approximation (Kleeorin & Rogachevskii 2018, this theory is incomplete)

A general expression for $\hat{T}$

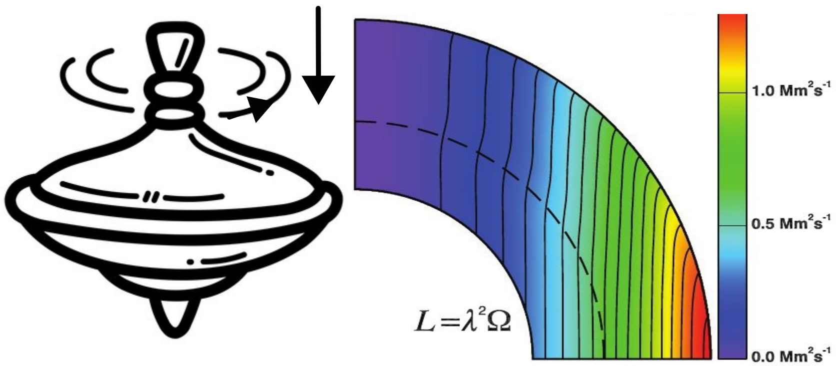
Application the mean-field hydrodynamic framework leads to the Taylor expansion in terms of the scale-separation parameter, ℓ / L_a:

$$\hat{T}_{\mathrm{ij}} = p_T \delta_{\mathrm{ij}} + \hat{T}^{(\Lambda)}_{\mathrm{ij}} + \hat{T}^{(\nu)}_{\mathrm{ij}} = p_T \delta_{\mathrm{ij}} + \Lambda_{\mathrm{ijk}} \Omega_k - \mathcal{N}_{\mathrm{ijkl}} \frac{\partial \langle U_k \rangle}{\partial x_l}$$

It is a sum of turbulent pressure, the non-dissipative (generation) momentum flux and eddy viscosity is a forth-rank tensor which dissipates the large-scale shear.

Azimuthal stresses and gyroscopic pumping

The correlation $\bar{\rho}\langle u_r u_\varphi\rangle$ is the angular momentum flux in radial direction, i.e., the azimuthal force along the radius, and $\bar{\rho}\langle u_\theta u_\varphi\rangle$ is angular momentum flux in meridional direction.



Julia

Here, we apply azimuthal force in vertical direction, so it an example of $\mathrm{Stress}_{z\varphi}$. On the Sun there is a similar process which pumps the angular momentum to equator. Gyroscopic pumping (Miecsh 2008)

General expression for Λ effect (Ruediger 1989)

We have $\langle u_i u_j \rangle = \Lambda_{\rm ijk} \Omega_k$, consider the case of slow rotation, i.e., we can keep the terms order of Ω^0 and Ω^2 (the even power!). From the symmetry and the reflection symmetry we can guess

- $\Lambda_{\rm ijk}$ is symmetric about i and j
- $\Lambda_{\rm ijk}$ is a pseudo-tensor i.e., 3-rank tensor which change sign under reflection

General expression for Λ effect (Ruediger 1989)

The only 3-rank pseudo-tensor is the tensor of Levi-Chevitta, $\varepsilon_{\rm ijk}$ hower it is antisymmetric. If there is another preferable direction, e.g., stratification, let us $\hat{\boldsymbol{g}}$ is a unit vector along the radius, then we can write a symmetric combination:

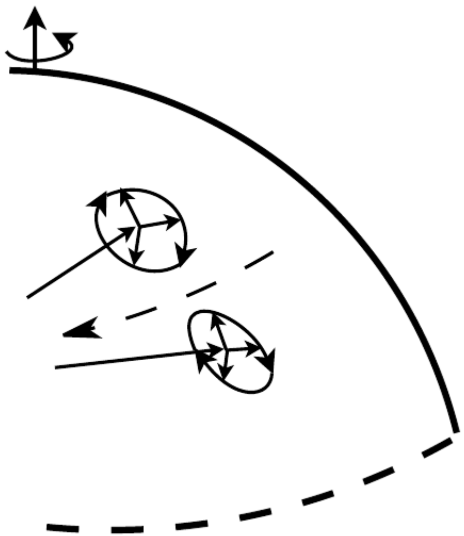
$$\Lambda_{\rm ijk} = A\left(\hat{g}_i \varepsilon_{\rm jnk} + \hat{g}_j \varepsilon_{\rm ink}\right)\hat{g}_n,$$

For the higher order term the combination is $\Lambda_{\rm ijk} = B(\hat{\boldsymbol{g}} \cdot \boldsymbol{\Omega})(\Omega_i \varepsilon_{\rm jnk} + \Omega_j \varepsilon_{\rm ink})\hat{g}_n$, then

$$\begin{aligned}
\langle u_r u_\varphi \rangle &= \Lambda_V \Omega \sin\theta = \Omega \sin\theta \left(A + B\Omega^2 \cos^2\theta\right) \\
\langle u_\theta u_\varphi \rangle &= \Lambda_H \Omega \cos\theta = -B\Omega^3 \sin^2\theta \cos\theta
\end{aligned}$$

Λ effect because of rotation and stratification

Kitchatinov (1987)



The net azimuthal flow (the dashed arrow) would be induced by cyclonic-type convection motions

$$\bar{\rho}\langle u_r u_\varphi\rangle = \Lambda_{r\varphi k}\Omega_k$$

The rising cells are rotating clockwise (looking from the pole), the sinking - anti-clockwise. Also those cyclons which are closer to the pole are rotating faster. Averaging over the rising blobs show the retrograde torque. The effect is proportional to $\Omega\,\tau\frac{\ell}{\bar{\rho}}\frac{\partial\,\bar{\rho}}{\partial\,r}$. The falling blobs gives the accelarating torque. The effect is

$$\Omega\sin\theta\,\frac{\left(\left(\tau\frac{\ell}{\bar{\rho}}\frac{\partial\,\bar{\rho}}{\partial\,r}\right)_R - \left(\tau\frac{\ell}{\bar{\rho}}\frac{\partial\,\bar{\rho}}{\partial\,r}\right)_F\right)}{\Delta r} \sim C\Omega\sin\theta\left(\frac{\partial}{\partial r}\frac{\tau\ell}{\bar{\rho}}\frac{\partial\,\bar{\rho}}{\partial\,r}\right)$$

Λ effect because of anisotropy of convection

$$\begin{aligned} u'_\phi &\approx -2\tau_c\Omega\sin\theta {u'_r}^{(0)} \\ u'_r &\approx 2\tau_c\Omega\sin\theta {u'_\phi}^{(0)} \\ \overline{u'_r {u'_\phi}^{(0)}} + \overline{u'_\phi {u'_r}^{(0)}} &\approx 2\tau_c\Omega\sin\theta\left(\overline{{u'_\phi}^{(0)2}} - \overline{{u'_r}^{(0)2}}\right) \end{aligned}$$

Quenching for the fast rotating regimes as $(2\tau_c\Omega)^{-3}$

Subsurface shear (Kitchatinov 2023)

For the stationary stage and the anisotropic background turbulence we get

$$\hat{T}_{r\varphi} = \Lambda_{r\varphi n}\Omega_n - \mathcal{N}_{r\varphi \mathrm{kl}}\frac{\partial \langle U_k \rangle}{\partial x_l} = 0;$$

$$\Omega(\langle u_h^2\rangle - 2\langle u_r^2\rangle) = \langle u_r^2\rangle r\frac{\partial \Omega}{\partial r};$$

$$\frac{r}{\Omega}\frac{\partial \Omega}{\partial r} = \frac{\langle u_h^2\rangle}{\langle u_r^2\rangle} - 2$$

Here, $\langle u_h^2\rangle = \langle u_\varphi^2\rangle + \langle u_\theta^2\rangle$. Shear is negative when $\langle u_r^2\rangle > \langle u_\varphi^2\rangle = \langle u_\theta^2\rangle$. Also, it is independent of latitude, as in the solar observations.

The meridional circulation is the poloidal part of the large-scale flow which satisfies the NS equations:

$$\bar{\rho}\partial_t\langle \boldsymbol{U}^p\rangle + \bar{\rho}(\langle \boldsymbol{U}^p\cdot\boldsymbol{\nabla}\rangle\langle \boldsymbol{U}^p\rangle \;=\; -\boldsymbol{\nabla}\bar{P} + \boldsymbol{g}\bar{\rho} + r\,\boldsymbol{e}^{\perp}\Omega^2 - \frac{\langle \boldsymbol{B}\rangle\times\boldsymbol{\nabla}\times\langle \boldsymbol{B}\rangle}{4\pi} - \boldsymbol{\nabla}\cdot\bar{\rho}\hat{\boldsymbol{T}}\,,$$

where $e^{\perp} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$ is the unit vector perpendecular to the axis of rotation. It is convenient to define the toroidal vorticity $\bar{\boldsymbol{\omega}} = (\boldsymbol{\nabla}\times\langle \boldsymbol{U}^{\boldsymbol{p}}\rangle)_{\varphi}$. Then we have

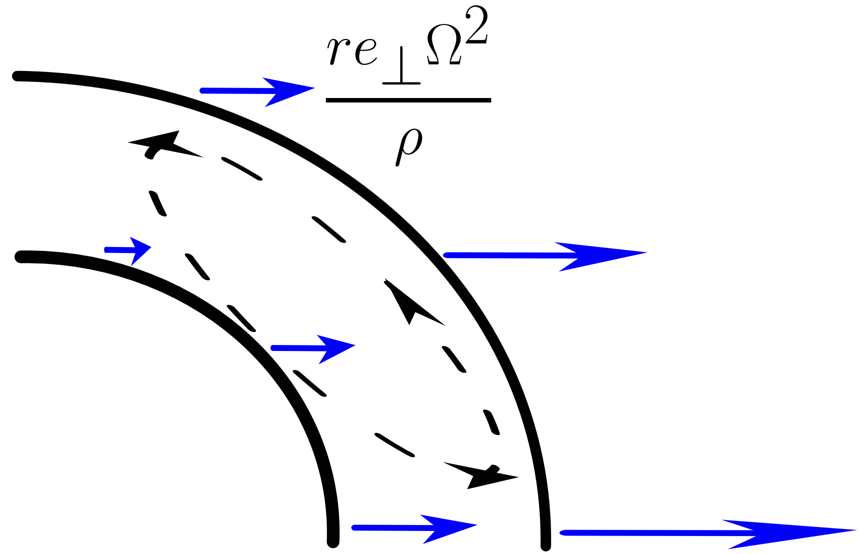
$$\begin{aligned}\frac{\partial\bar{\omega}}{\partial t} + \mathrm{rsin}\theta\boldsymbol{\nabla}\cdot\frac{\langle \boldsymbol{U}^p\rangle\bar{\omega}}{\mathrm{rsin}\theta} \;&=\; \mathrm{rsin}\theta\boldsymbol{\nabla}\cdot\frac{\hat{\boldsymbol{\varphi}}\times\boldsymbol{\nabla}\,\bar{\rho}\hat{\boldsymbol{T}}}{r\bar{\rho}\mathrm{sin}\theta}(\mathrm{dissipation})\\ &+\; \mathrm{rsin}\theta\frac{\partial\Omega^2}{\partial z} - \frac{g}{c_p r}\frac{\partial\bar{s}}{\partial\theta}(\mathrm{generation})\\ &+\; \boldsymbol{\nabla}\cdot\frac{1}{\bar{\rho}}(\langle \boldsymbol{B}\rangle\langle J_{\varphi}\rangle - \langle \boldsymbol{J}\rangle\langle B_{\varphi}\rangle)(\mathrm{quenching})\end{aligned}$$

where $\langle \boldsymbol{J}\rangle = \frac{1}{4\pi}\boldsymbol{\nabla}\times\langle \boldsymbol{B}\rangle$

Sources of the meridional circulation

Centrifugal force. For example, the equator rotates faster than the pole the density force $\frac{r\boldsymbol{e}_{\perp}\Omega^2}{\bar{\rho}}$

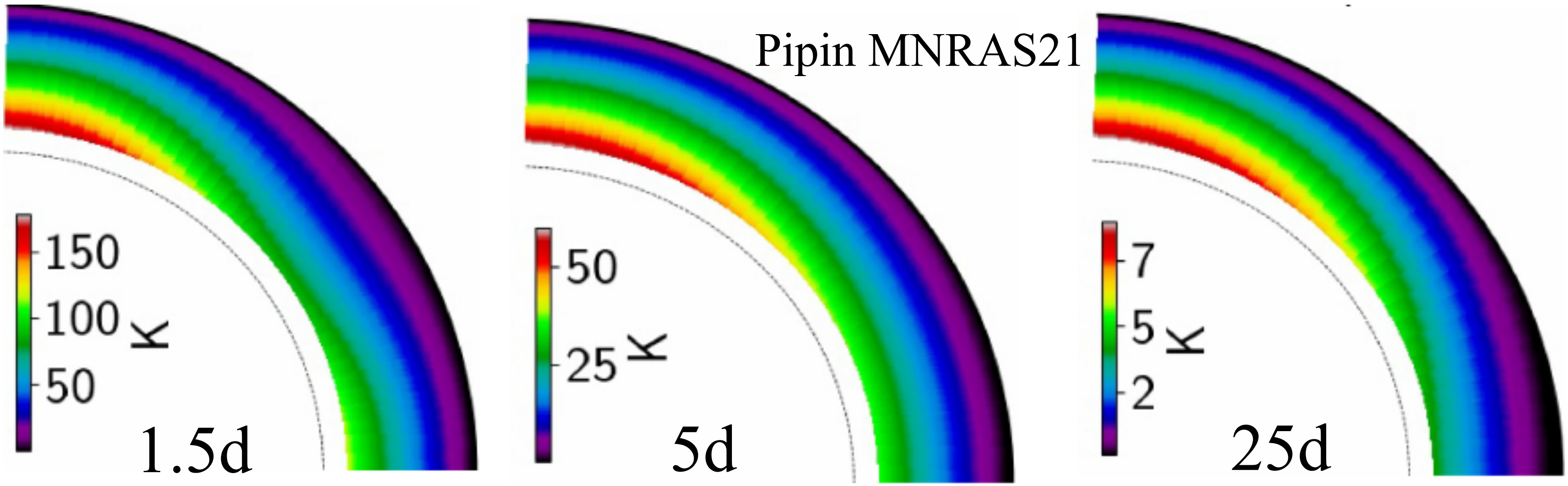
is large at the top of CZ equator and small in polar regions. Therefore it initiate anti-clockwise circulation to compensate the redistribution of the angular momentum



Thermal wind (baroclinic force). It comes from $\nabla \times \frac{1}{\bar{\rho}} \nabla \bar{P} = \nabla(\frac{1}{\bar{\rho}}) \times \nabla \bar{P}$. Also we employ the hydrostatic condition $\boldsymbol{\nabla} \bar{P} = -\boldsymbol{g} \bar{\rho}$, neglect asphericity, and use the equation of state: $\dfrac{\boldsymbol{\nabla} \bar{\rho}}{\bar{\rho}} = -\dfrac{\boldsymbol{\nabla} \bar{s}}{c_p} + \dfrac{\boldsymbol{\nabla} \bar{P}}{\overline{\gamma P}}$. Finally:

$$\left(\nabla(\frac{1}{\bar{\rho}}) \times \nabla \bar{P} \right)_{\varphi} = -\frac{1}{\bar{\rho}^2} (\nabla \bar{\rho} \times \nabla \bar{P})_{\varphi} = -\frac{g}{c_p} \frac{\partial \bar{s}}{\partial \theta}$$

In rotating convective zone the heat-flux is anisotropic. This resuts to the warm pole phenomena. Because of huge heat capacity of the SCZ, the effect is screened at the surface. There, the temperature difference between pole and equator it is no more than 1K.


Pipin MNRAS21
150
100
50
K
1.5d
50
25
K
5d
7
5
2
K
25d

The Taylor-Proudman balance

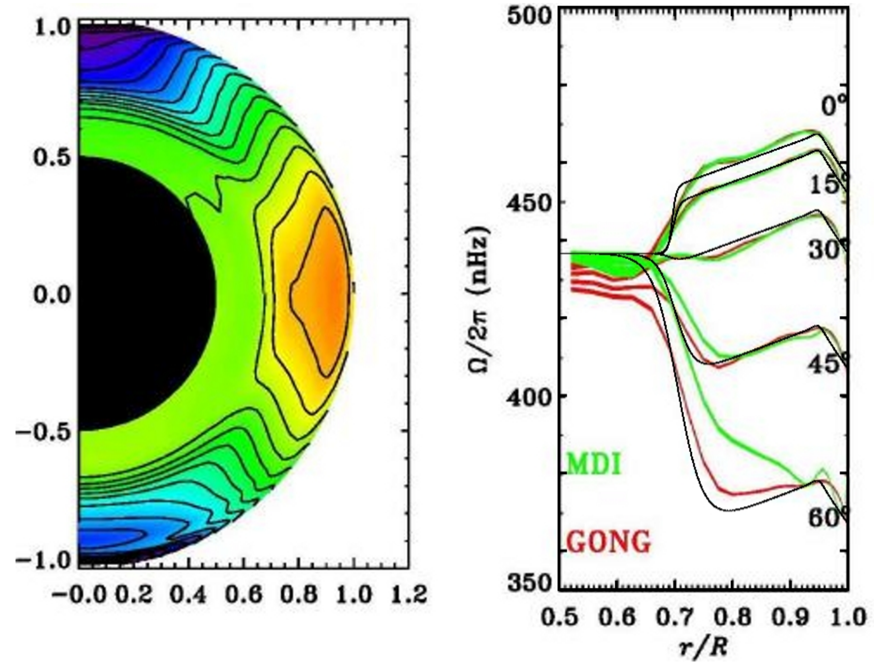
Note that

$$r\sin\theta\frac{\partial\Omega^2}{\partial z}-\frac{g}{c_p r}\frac{\partial\bar{s}}{\partial\theta}\approx 0$$

Corresponds to the Taylor-Proudman balance, when the flow is uniform along the rotation axis,

$$2\Omega\times\langle U\rangle \;\approx\; -\frac{\mathbf{1}}{\bar{\boldsymbol{\rho}}}\boldsymbol{\nabla}\bar{P}$$

Consider $\langle \boldsymbol{U}\rangle=\boldsymbol{e}_{\varphi}r\sin\theta\Omega$, then, after curling the above equation, we get $r\sin\theta\frac{\partial\Omega^2}{\partial z}-\frac{g}{c_p r}\frac{\partial\bar{s}}{\partial\theta}\approx 0$. For the rotating CZ, the TP balance mean that the angular velocity does not vary along the rotation axis. The helioseismology finds:



This means that the SCZ shows deviations from Taylor-Proudman balance.

Note that for the rotating star

$$\boldsymbol{\nabla}\bar{\rho}\times\boldsymbol{\nabla}\bar{P}\neq 0$$

The basic equations for the large-scale flow

$$\frac{\partial}{\partial t}\bar{\rho}r^2\sin^2\theta\Omega \;=\; -\boldsymbol{\nabla}\cdot\left(r\sin\theta\left(\bar{\rho}\hat{\mathbf{T}}_\phi + \bar{\rho}\, r\,\sin\theta\Omega\langle\boldsymbol{U}\rangle - \frac{\langle\boldsymbol{B}\rangle\langle B_\phi\rangle}{4\pi}\right)\right)$$

where $\hat{T}_{\rm ij} \;=\; \langle u_i u_j\rangle - \frac{1}{4\pi\bar{\rho}}(\langle b_i b_j\rangle - \delta_{\rm ij}\langle b^2\rangle)$, the mean-field heat-transport equation:

$$\bar{\rho}\bar{T}\left(\frac{\partial\langle s\rangle}{\partial t} + (\langle\boldsymbol{U}\rangle\cdot\boldsymbol{\nabla})\langle s\rangle\right) \;=\; -\boldsymbol{\nabla}\cdot(\mathbf{F}^c+\mathbf{F}^{\rm rad}) - \hat{T}_{\rm ij}\frac{\partial\langle U_i\rangle}{\partial r_j} - \boldsymbol{\mathcal{E}}\cdot\boldsymbol{\nabla}\times\langle\boldsymbol{B}\rangle$$

where the effect of turbulent convection, $F_i^c \;=\; -c_p\bar{\rho}\bar{T}\kappa_{\rm ij}\nabla_j\langle s\rangle, \kappa_{\rm ij} = \langle u_i u_j\rangle$.

$$\frac{\partial\bar{\omega}}{\partial t} + {\rm rsin}\theta\boldsymbol{\nabla}\cdot\frac{\langle\boldsymbol{U}^p\rangle\bar{\omega}}{{\rm rsin}\theta} \;=\; {\rm rsin}\theta\boldsymbol{\nabla}\cdot\frac{\hat{\varphi}\times\boldsymbol{\nabla}\bar{\rho}\hat{\boldsymbol{T}}}{r\bar{\rho}{\rm sin}\theta} + {\rm rsin}\theta\frac{\partial\Omega^2}{\partial z} - \frac{g}{c_p r}\frac{\partial\bar{s}}{\partial\theta}$$

$$+ \;\; \boldsymbol{\nabla}\cdot\frac{1}{\bar{\rho}}(\langle\boldsymbol{B}\rangle\langle J_\varphi\rangle - \langle\boldsymbol{J}\rangle\langle B_\varphi\rangle)$$

The boundary conditions

The top of the convection zone stress-free and the black-body radiation:

$$\bar{\rho}\hat{\mathrm{T}}_{r\varphi}-\frac{\langle B_r\rangle\langle B_\varphi\rangle}{4\pi}=0;\hat{\mathrm{T}}_{r\theta}=\omega-\frac{2\langle U_\theta\rangle}{r}=0;\mathrm{F}_r^c+\mathrm{F}_r^{\mathrm{rad}}=\frac{L_\odot}{4\pi r_t^2}\left(1+\left(\frac{\bar{s}}{c_p}\right)^4\right),r_t=0.99R_\odot$$

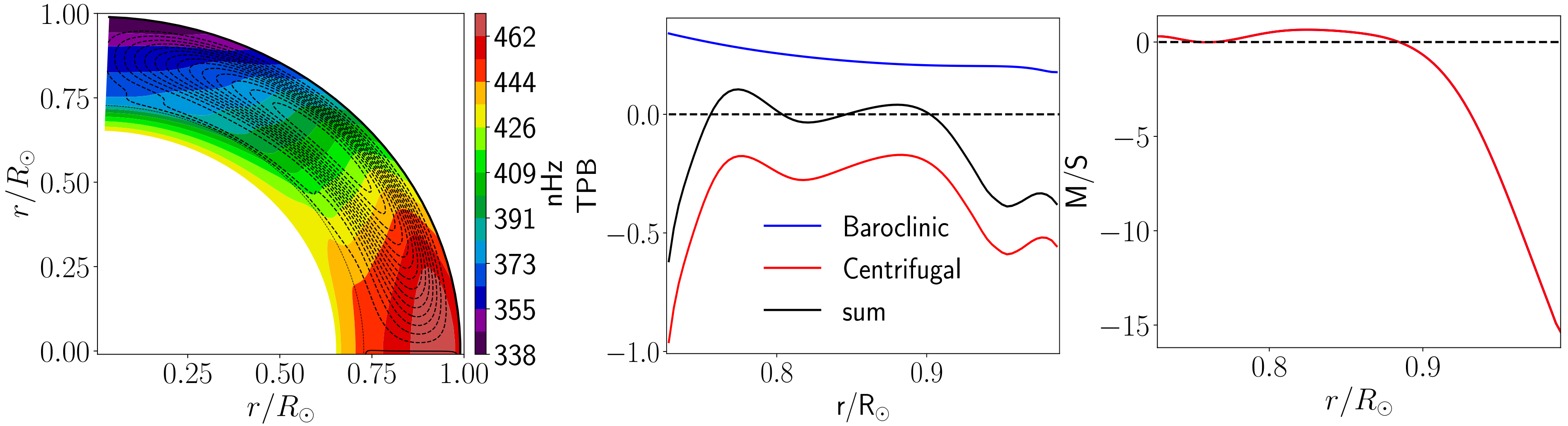
We put the bottom of the convection zone at $r_b=0.728R_\odot$, $\mathrm{F}_r^c+\mathrm{F}_r^{\mathrm{rad}}=\frac{L_\odot}{4\pi r_b^2}$. The rigid rotation is at $r_i=0.65R_\odot$.

The mixing length approximation:

$$\ell_c=\alpha_{\mathrm{MLT}}H_p, H_p=(\partial_r\log\bar{P})^{-1}; \langle u^2\rangle=-\frac{\ell_c^2 g}{8c_p}\frac{\partial\bar{s}}{\partial r}$$

The thermodynamic reference state of the convection zone is from MESA code(http://mesa.sourceforge.net/)

The mean-field model

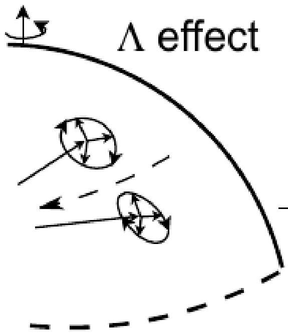


Results from Pipin&Kosovichev (2019) nonmagnetic case. The angular momentum distribution results from a balance of the meridional circulation and the turbulent stresses.

The stage of the bulk convection zone is close to the Taylor-Proudman balance. The deviations from the TP balance are concentrated to the boundaries. At the bottom - transition to the tachocline and at the top - stress-free boundary conditions exclude the local TP balance

Summary

(Kithchatinov 2011, Miesch 2016)



Ω +convection

Differential rotation

Meridional circulation

$$\sin\theta\, r \frac{\partial \Omega^2}{\partial z} - \frac{g}{c_\mathrm{p} r}\frac{\partial S}{\partial \theta} \approx 0$$

Taylor-Proudman balance

Anisotropic convection heats up poles

“Thermal wind”